Approximation Theory of Deep Neural Networks

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Deep neural networks form the foundation of an extremely successful modern machine learning method called deep learning. To understand the success of deep learning it is therefore essential to characterise the capabilities and the limitations of this architecture. One of the most fundamental questions in this regard is to identify the classes of functions that can be constructed as neural networks with a given architecture, or at least approximated up to a certain accuracy. Approximation theory of deep neural networks studies precisely this problem. In this lecture series, we will showcase hallmark results in the area of approximation theory of neural networks that yield substantial insight into the effect of the choice of network architecture.

The following topics will be covered in one lecture each.

1. *Classical Approximation Theory of Neural Networks:* We start by defining precisely neural networks as well as the associated approximation problem. The famous universal approximation theorem as well as quantitative approximation of smooth functions will be discussed. In addition, we will study the connection the the Kolmogorov-Arnold superposition theorem.

2. *Deep ReLU Neural Networks:* The most widely used activation function in practice is the so-called ReLU. This function is piecewise linear with two pieces. Nonetheless, it is possible to approximate non-linear functions in a very efficient way by deep ReLU neural networks. We will describe this theory, as well as connections to finite element approximation.

3. *The Role of Depth:* One central step towards understanding the success of deep learning is to understand the role of depth. Deep neural networks appear to outperform their shallow counterparts. From an approximation theoretical point of view there are some possible explanations, which we discuss in this part of the lecture series.

4. *The Curse of Dimension:* High dimensional problems are typically plagued by the curse of dimension, i.e., the exponential deterioration of approximation quality with growing dimension. In practice, such as in image classification, this effect is hardly observable. In this part of the lecture series, we investigate reasons for this. This includes hierarchy assumptions, the famous manifold hypothesis, as well as, Barron spaces.

5. *Lower Bounds on Approximation Rates:* We finish the lecture series by establishing lower bounds on the best achievable approximation rates. We will collect different methods from the literature and show that many of the previously observed bounds are optimal under very weak assumptions that should always be satisfied in practice. We will also find that some approximation rates in the literature surprisingly beat this optimality which implies that they must be impractical.