Transport along stochatic flows in fluid dynamics

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Ensemble forecasting and filtering are widely used in geophysical sciences for numerical weather forecasting and climate projection application. In practice to be efficient these methods require an accurate physical modeling of the dynamical model errors. These errors evolve along time and strongly interact with the large-scale state variables of interest. The generic design of large-scale geophysical models incorporating errors or uncertainty is consequently far from being an easy task. To address this issue, we propose as initiated by [1] to model the fluid flow by a continuous semi-martingale $\mathrm{d}\boldsymbol{X}_t = \mathbf{w}\mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t$. Within this simple assumption and Ito-Wentzell formula ([2]), the material derivative, – the derivative along the flow trajectory $\boldsymbol{X}_t$ – of a tracer, Θ, has to be modified as:

$$0 = \left(\mathrm{d}\left[\Theta\left(t, \boldsymbol{X}_t\right)\right]\right)_{|\boldsymbol{X}_t = \boldsymbol{x}} = \underbrace{\mathrm{d}_t\Theta}_{\substack{\text{Time}\\ \text{increment}}} + \underbrace{\mathbf{w}^* \cdot \boldsymbol{\nabla}\Theta}_{\text{Advection}}\mathrm{d}t + \underbrace{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot \boldsymbol{\nabla}\Theta}_{\text{Noise}} - \underbrace{\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)}_{\text{Diffusion}}\mathrm{d}t, \tag{1}$$

with, $\mathbf{w}^* = \mathbf{w} - \frac{1}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{a})^T$, an effective drift and, $\boldsymbol{a}\mathrm{d}t = \mathrm{d}\left\langle \boldsymbol{X}, \boldsymbol{X}^T \right\rangle$, the quadratic covariation matrix of the flow. Compared to a usual transport equation, three new terms appear in this expression: (*i*) a drift correction, (*ii*) an inhomogeneous and anisotropic diffusion and (*iii*) a multiplicative noise. These three terms are strongly linked together, which ensures desired properties such as energy conservation.

With this stochastic version of the transport equation, it is possible to express the fundamental conservation laws of classical mechanics and to derive stochastic versions of *a priori* any fluid dynamics models. Following this procedure, we have derived and simulated a stochastic version of the Surface Quasi-Geostrophic (SQG) model. We have shown that the realizations of this stochastic version allows us to better resolve the small-scales in comparizon to the usual SQG model. Besides, we have evidenced that an ensemble of realization was able to accurately estimate at each time step the amplitudes and positions of the model errors in both spatial and spectral domains. In comparison a classical randomization of the initial state leads to an underestimation of one order of magnitude. Our ensemble also succeeded to predict density skewness and extreme events of the tracer at small scales. The presentation will explicit our stochastic version of the material derivative and comment the numerical results obtained.

References

[1] Mémin, E. (2014) *Fluid flow dynamics under location uncertainty* , Geophysical & Astrophysical Fluid Dynamics 108.2, 119-146.

[2] Kunita, H. (1997) Stochastic flows and stochastic differential equations, Vol. 24. Cambridge university press.