Introduction to random fields and scale invariance: Lecture II

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Outlines

Lecture 2 :

1. Sample paths regularity
   1. Hölder regularity
   2. Critical Hölder exponent
   3. Directional Hölder regularity
2. Hausdorff dimension of graphs
   1. Hausdorff measures and dimensions
   2. Upper bound of graphs Hausdorff dimension
   3. Lower bound of graphs Hausdorff dimension

Stochastic continuity and modification

Definition

Let $X = (X_x)_{x\in\mathbb{R}^d}$ be a random field. We say that X is stochastically continuous at point $x_0 \in \mathbb{R}^d$ if

$$\forall \varepsilon > 0, \lim_{x\to x_0} \mathbb{P}(|X_x - X_{x_0}| > \varepsilon) = 0.$$

Definition

Let $X = (X_x)_{x\in\mathbb{R}^d}$ be a random field. We say that $\tilde{X} = (\tilde{X}_x)_{x\in\mathbb{R}^d}$ is a modificiation of X if

$$\forall x \in \mathbb{R}^d, \mathbb{P}(X_x = \tilde{X}_x) = 1.$$

Remark : It follows that X and $\tilde{X}$ have the same law.

Regularity

Definition

Let $K = [0,1]^d$. *Let* $\gamma \in (0,1)$. *A random field* $X = (X_x)_{x\in\mathbb{R}^d}$ *is* γ*-Hölder on* K *if there exists a rv* A *such that a.s.*

$$|X(x) - X(y)| \leq A\|x-y\|^\gamma, \forall x, y \in K.$$

Theorem (Kolmogorov-Chentsov 1956)

If there exist $0 < \beta < \delta$ *and* $C > 0$ *such that*

$$\mathbb{E}\left(|X(x) - X(y)|^\delta\right) \leq C\|x-y\|^{d+\beta}, \forall x, y \in K,$$

then there exists $\tilde{X}$ *a modification of* X γ*-Hölder on* K*, for all* $\gamma < \beta/\delta$.

Proof of Theorem Step 1

For $k \geq 1$ we introduce the dyadic points of $[0,1]^d$

$$\mathcal{D}_k = \left\{ \frac{j}{2^k}; \forall 1 \leq i \leq d, 0 \leq j_i \leq 2^k \right\}.$$

Note that for $x \in [0,1]^d$, $\exists x_k \in \mathcal{D}_k$ with $\|x - x_k\|_\infty \leq 2^{-k}$.
Let $\gamma \in (0, \beta/\delta)$. For $i, j \in [0, 2^k]^d \cap \mathbb{N}^d$ with $i \neq j$ define

$$E^k_{i,j} = \{\omega \in \Omega; |X_{i/2^k}(\omega) - X_{j/2^k}(\omega)| > \|i/2^k - j/2^k\|^\gamma_\infty\}.$$

By assumption and Markov inequality

$$\mathbb{P}(E^k_{i,j}) \leq 2^{-k(d+\beta-\gamma\delta)} \|i - j\|_\infty^{d+\beta-\gamma\delta}.$$

Proof of Theorem Step 1

Let set

$$E^k = \bigcup_{(i,j)\in[0,2^k];0<\|i-j\|_\infty\leq 5} E^k_{i,j}.$$

It follows that

$$\begin{aligned}\mathbb{P}(E^k) &\leq 5^{d+\beta-\gamma\delta}2^{-k(d+\beta-\gamma\delta)}\#\{(i,j)\in[0,2^k];0<\|i-j\|_\infty\leq 5\} \\ &\leq 5^{d+\beta-\gamma\delta}10^d 2^{-k(\beta-\gamma\delta)}\end{aligned}$$

Hence, by Borel Cantelli $\mathbb{P}(\limsup E^k) = 0$ and $\tilde{\Omega} = \cup_k \cap_{l\geq k} E^l$ is such that $\mathbb{P}(\tilde{\Omega}) = 1$. For $\omega \in \tilde{\Omega}$, $\exists k^*(\omega)$ s.t. $\forall l \geq k^*(\omega)$, $x, y \in \mathcal{D}_l$ with $0 < \|x - y\|_\infty \leq 52^{-l}$

$$|X_x(\omega) - X_y(\omega)| \leq \|x - y\|^\gamma_\infty.$$

Proof of Theorem Step 2

Let us set $\mathcal{D} = \cup_k \mathcal{D}_k$. For $x, y \in \mathcal{D}$ with $0 < \|x - y\|_\infty \leq 2^{-k^*(\omega)}$, $\exists! l \geq k^*(\omega)$ with

$$2^{-(l+1)} < \|x - y\|_\infty \leq 2^{-l}.$$

Moreover, one can find $n \geq l + 1$ s.t. $x, y \in \mathcal{D}_n$ and $\forall k \in [l, n-1]$, $\exists x_k, y_k \in \mathcal{D}_k$ with $\|x - x_k\|_\infty \leq 2^{-k}$ and $\|y - y_k\|_\infty \leq 2^{-k}$. We set $x_n = x$ and $y_n = y$. Therefore

$$\begin{aligned} \|x_l - y_l\|_\infty &\leq \|x_l - x\|_\infty + \|x - y\|_\infty + \|y - y_l\|_\infty \\ &\leq 22^{-l} + \|x - y\|_\infty \end{aligned}$$

But $2^{-l} < 2\|x - y\|_\infty$ and $\|x_l - y_l\|_\infty \leq 5\|x - y\|_\infty \leq 52^{-l}$ and since $l \geq k^*(\omega)$

$$|X_{x_l}(\omega) - X_{y_l}(\omega)| \leq \|x_l - y_l\|_\infty^\gamma \leq 5^\gamma \|x - y\|_\infty^\gamma.$$

Proof of Theorem Step 2

But for all $k \in [l, n-1]$,
$\|x_k - x_{k+1}\|_\infty \leq 2^{-k} + 2^{-(k+1)} \leq 32^{-(k+1)}$ s.t.

$$|X_{x_k}(\omega) - X_{x_{k+1}}(\omega)| \leq \|x_k - x_{k+1}\|_\infty^\gamma \leq (3/2)^\gamma 2^{-k\gamma}.$$

Similarly,

$$|X_{y_k}(\omega) - X_{y_{k+1}}(\omega)| \leq \|y_k - y_{k+1}\|_\infty^\gamma \leq (3/2)^\gamma 2^{-k\gamma}.$$

It follows that

$$\begin{aligned} |X_x(\omega) - X_y(\omega)| &\leq \sum_{k=l}^{n-1} |X_{x_k}(\omega) - X_{x_{k+1}}(\omega)| + |X_{x_l}(\omega) - X_{y_l}(\omega)| \\ &+ \sum_{k=l}^{n-1} |X_{y_k}(\omega) - X_{y_{k+1}}(\omega)| \\ &\leq \frac{2 \times 3^\gamma}{2^\gamma - 1} \times 2^{-l\gamma} + 5^\gamma \|x - y\|_\infty^\gamma \\ &\leq c_\gamma \|x - y\|_\infty^\gamma \end{aligned}$$

Proof of Theorem Step 3

By chaining, it follows that $\forall x, y \in \mathcal{D}$

$$|X_x(\omega) - X_y(\omega)| \leq c_\gamma 2^{k^*(\omega)} \|x - y\|_\infty^\gamma,$$

and we set $A(\omega) = c_\gamma 2^{k^*(\omega)}$. Hence we have proven that $\forall \omega \in \tilde{\Omega}$, $x, y \in \mathcal{D}$,

$$|X_x(\omega) - X_y(\omega)| \leq A(\omega) \|x - y\|_\infty^\gamma.$$

We set $\tilde{X}_x(\omega) = 0$ if $\omega \notin \tilde{\Omega}$. For $\omega \in \tilde{\Omega}$, if $x \in \mathcal{D}$ we set $\tilde{X}_x(\omega) = X_x(\omega)$. Otherwize, there exists $(x_k)_k$ a sequence of dyadic points such that $x_k \to x$. Therefore $(X_{x_k}(\omega))$ is a Cauchy sequence and we define $\tilde{X}_x(\omega)$ as its limit. By stochastic continuity we have

$$\mathbb{P}(\tilde{X}_x = X_x) = 1.$$

Critical Hölder exponent

Definition

Let $\gamma \in (0,1)$. A random field $(X(x))_{x\in\mathbb{R}^d}$ admits γ as critical Hölder exponent on $[0,1]^d$ if :

(a) $\forall s < \gamma$, *a.s.* X *satisfies* $H(s)$: $\exists A \geq 0$ *rv s.t.* $\forall x, y \in [0,1]^d$,

$$|X(x) - X(y)| \leq A\|x - y\|^s.$$

(b) $\forall s > \gamma$, *a.s.* X *fails to satisfy* $H(s)$.

Proposition (Adler, 1981)

Let $(X(x))_{x\in\mathbb{R}^d}$ be a Gaussian random field. If $\forall \delta > 0$, $\exists c_1, c_2 > 0$,

$$c_1\|x - y\|^{2\gamma+\delta} \leq \mathbb{E}\left(X(x) - X(y)\right)^2 \leq c_2\|x - y\|^{2\gamma-\delta}$$

➥ *critical Hölder exponent on $[0,1]^d = \gamma$ for any continuous version of X.*

Gaussian s.i. fields

Corollary

For X s.i. Gaussian s.t. $\forall \delta > 0$, $\exists c_1, c_2 > 0$,

$$c_1 \|x\|^{2\gamma+\delta} \leq v_X(x) = \mathbb{E}((X_x - X_0)^2) \leq c_2 \|x\|^{2\gamma-\delta},$$

any continuous version of X have critical Hölder exponent on $[0,1]^d = \gamma$.

Exples :

- FBF with $v_H(x) = \|x\|^{2H}$ and EFBF $v_{H,\alpha}(x) = c_{H,\alpha}(x/\|x\|)\|x\|^{2H}$
  critical Hölder exponent $= H$
- OU $v(t) = 2\,(c(0) - c(t)) = 2(1 - e^{-\theta|t|})$
  critical Hölder exponent $= 1/2$
- OS $v_{H,E}(x) = \left(\sum_{i=1}^d |\langle x, \theta_i \rangle|^{2\alpha_i}\right)^H$
  critical Hölder exponent $= H \min_{1\leq i\leq d} \alpha_i$

Directional Hölder regularity

Let $x_0 \in \mathbb{R}^d$ and $\theta \in S^{d-1}$, the line process $L_{x_0,\theta}(X) = (X(x_0 + t\theta))_{t\in\mathbb{R}}$ has variogram $v_\theta(t) = \mathbb{E}\left((X(x_0 + t\theta) - X(x_0))^2\right) = v_X(t\theta)$.

Definition (Bonami, Estrade, 2003)

Let $\theta \in S^{d-1}$. We say that X admits $\gamma(\theta) \in (0,1)$ as directional regularity in the direction θ if $\forall \delta > 0$, $\exists c_1, c_2 > 0$,

$$c_1 |t|^{2\gamma(\theta)+\delta} \leq v_\theta(t) = v_X(t\theta) \leq c_2 |t|^{2\gamma(\theta)-\delta},$$

Proposition (Bonami, Estrade, 2003)

If $\exists \gamma : S^{d-1} \to (0,1)$ s.t. $\forall \theta \in S^{d-1}$, X admits $\gamma(\theta)$ as directional regularity in the direction θ. Then γ takes at most d values. Moreover, if γ takes k values $\gamma_k < \ldots < \gamma_1$, there exist

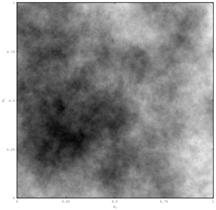
$$\{0\} = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_k := \mathbb{R}^d$$

$$\gamma(\theta) = \gamma_i \Leftrightarrow \theta \in (V_i \setminus V_{i-1}) \cap S^{d-1}.$$

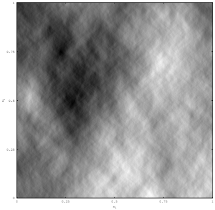
Directional Hölder regularity

Exples :

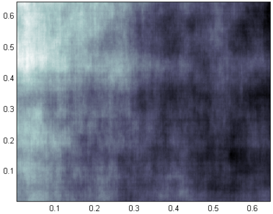
- FBF with $v_H(x) = \|x\|^{2H}$ and EFBF $v_H(x) = c_{H,\alpha}(x/\|x\|)\|x\|^{2H}$: $\forall \theta \in S^{d-1}$, directional Hölder regularity in direction θ = H
- OS $v(x) = \left(\sum_{i=1}^{d} |\langle x, \theta_i \rangle|^{2\alpha_i}\right)^H$: $\forall 1 \leq i \leq d$, directional Hölder regularity in direction $\tilde{\theta}_i$ = $H\alpha_i$ for $E\tilde{\theta}_i = \alpha_i^{-1}\tilde{\theta}_i$



$H = 0.5$ FBF



$H = 0.5,\ \alpha = \pi/3$, EFBF



$H\alpha_1 = 0.5,\ H\alpha_2 = 0.6,\ H = 0.7$

Hausdorff measures and dimensions

Definition

Let $U \subset \mathbb{R}^d$ bounded. For $\delta > 0$ and $s \geq 0$, we set

$$\mathcal{H}^s_\delta(U) = \inf \left\{ \sum_{i \in I} diam(B_i)^s ; (B_i)_{i \in I} \ \delta - covering \ of \ U \right\},$$

meaning that $U \subset \cup_{i \in I} B_i$ with $diam(B_i) \leq \delta$.

Note that $\forall \delta < \delta'$, $\mathcal{H}^s_\delta(U) \geq \mathcal{H}^s_{\delta'}(U)$.

Definition

The s-dimensional Hausdorff measure of U is defined by

$$\mathcal{H}^s(U) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(U) \in [0, +\infty],$$

$s \mapsto \mathcal{H}^s(U)$ jumps from $+\infty$ to 0 : Hausdorff dimension of U

$$dim_H(U) = \inf \{ s \geq 0 ; \mathcal{H}^s(U) = 0 \} = \sup \{ s \geq 0 ; \mathcal{H}^s(U) = +\infty \}.$$

Hausdorff measures and dimensions

Remark : in general we do not know the value of $\mathcal{H}^{s^*}(U) \in [0, +\infty]$ at $s^* = \dim_H(U)$. But

$$\begin{aligned} \mathcal{H}^s(U) > 0 \quad &\Rightarrow \quad \dim_H(U) \geq s \\ \mathcal{H}^s(U) < +\infty \quad &\Rightarrow \quad \dim_H(U) \leq s \end{aligned}$$

Exple : $U = [0,1]^d$, $\|\cdot\|_\infty$, $U \subset \cup_{i=1}^{N_\delta} B_i(\delta)$ and $\mathcal{H}^s_\delta(U) \leq c\delta^{s-d}$.
It follows that for $s \geq d$, $\mathcal{H}^s(U) < +\infty$ and $\dim_H(U) \leq d$.
Let $U \subset \cup_{i\in I} B_i \subset \cup_{i\in I} B_i(r_i)$ and $1 = \mathcal{L}eb(U) \leq \sum_i r_i^d$ s.t. $\mathcal{H}^d(U) > 0$ and $\dim_H(U) = d$.

Proposition

If U is a non-empty open bounded set of $\mathbb{R}^d$ then $\dim_H(U) = d$.

Hausdorff dimensions for Hölder functions

Let $f : [0,1]^d \to \mathbb{R}$ and write

$$\mathcal{G}_f = \{(x, f(x)); x \in [0,1]^d\} \subset \mathbb{R}^{d+1}.$$

Note that $\dim_H \mathcal{G}_f \geq d$.

Proposition

If $|f(x) - f(y)| \leq C\|x - y\|_\infty^\gamma$ for $\gamma \in (0,1]$, then $\dim_H \mathcal{G}_f \leq d + 1 - \gamma$.

Write $[0,1]^d \subset \bigcup_{i=1}^{N_\delta} x_i + [0,\delta]^d$, then

$$\begin{aligned} \mathcal{G}_f &\subset \bigcup_{i=1}^{N_\delta} \left(x_i + [0,\delta]^d\right) \times \left(f(x_i) + [-C\delta^\gamma, C\delta^\gamma]\right) \\ &\subset \bigcup_{i=1}^{N_\delta} \bigcup_{j=1}^{N_\delta^\gamma} \left(x_i + [0,\delta]^d\right) \times \left(f(x_i) + I_j(\delta)\right). \end{aligned}$$

Hence $\mathcal{H}_\delta^s(\mathcal{G}_f) \leq N_\delta N_\delta^\gamma \delta^s \leq c\delta^{-d+\gamma-1+s}$. Therefore $s > d + 1 - \gamma$ implies $\mathcal{H}^s(\mathcal{G}_f) = 0$ and $\dim_H \mathcal{G}_f \leq d + 1 - \gamma$.

Frostman criteria for random fields

Theorem

Let $(X_x)_{x\in\mathbb{R}^d}$ *be a second order field a.s. continuous on* $[0,1]^d$ *s.t.* $\exists s>1$,

$$\int_{[0,1]^d\times[0,1]^d} \mathbb{E}\left(\left(|X_x - X_y|^2 + \|x-y\|^2\right)^{-s/2}\right) dxdy < +\infty,$$

then a.s. $dim_H \mathcal{G}_X \geq s$.

Corollary

Let $(X(x))_{x\in\mathbb{R}^d}$ *be a Gaussian random field. If* $\forall\delta>0$, $\exists c_1, c_2>0$,

$$c_1\|x-y\|^{2\gamma+\delta} \leq \mathbb{E}\left(X(x)-X(y)\right)^2 \leq c_2\|x-y\|^{2\gamma-\delta},$$

for any continuous modification $\tilde{X}$ *of* X

$$\Rightarrow dim_H \mathcal{G}_{\tilde{X}} = d+1-\gamma \text{ a.s}$$

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