Calabi-Yau categories, string topology, and Floer field theory

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Report on joint work with Sheel Ganatra

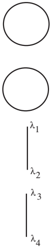
Proof of a conjecture (C., Schwarz, Cielebak - Latchev, Eliashberg) from 2003 relating two 2D topological field theories:

- The string topology of a closed oriented manifold M,
- The Floer - symplectic field theory of its cotangent bundle T^*M.

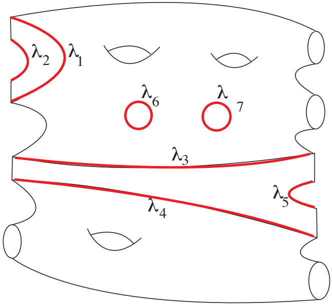
What is a 2D (open-closed) Topological Field Theory (TFT)?

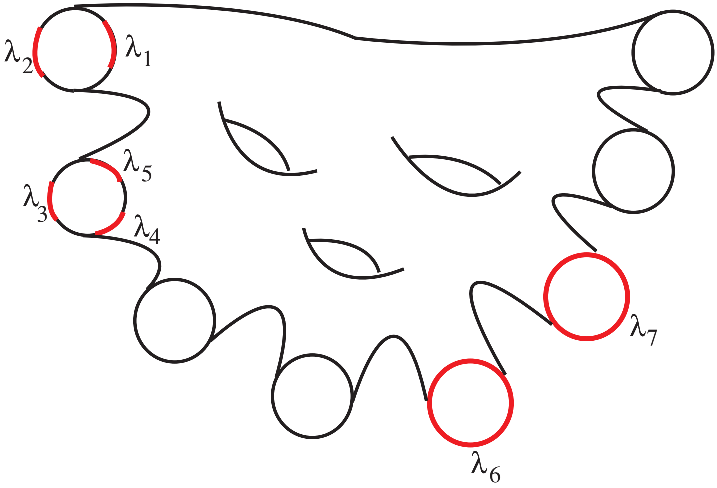
Axioms given by Atiyah, Segal, Witten additions by Moore-Segal, Kontsevich, Lurie

Topologically, the objects of study in such a field theory $\mathcal{F}$ are compact, oriented one-manifolds, c, together with a labeling of the boundary endpoints by elements of a set, $\mathcal{D}$ (= "D-branes" in physical examples). It also studies 2D cobordisms between them.



An "open-closed" cobordism Σ_{c_1, c_2} between two objects c_1 and c_2 is an oriented surface Σ, whose boundary is partitioned into three parts: the incoming boundary, $\partial_{in}\Sigma$ which is identified with c_1, the outgoing boundary $\partial_{out}\Sigma$ which is identified with c_2, and the remaining part of the boundary, referred to as the "free part", $\partial_{free}\Sigma$




λ_1
λ_2
λ_3
λ_4
λ_5
λ_6
λ_7

A TFT $\mathcal{F}$ assigns to an object c an algebraic object like a vector space (Hilbert) or a chain complex $\mathcal{F}(c)$.

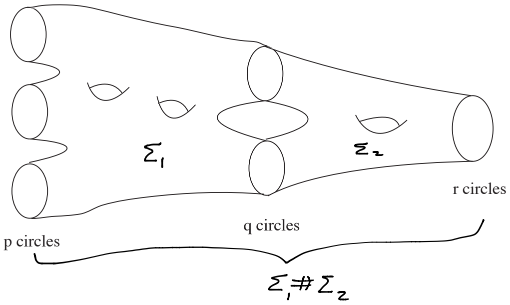
If Σ_{c_1,c_2} is an open-closed cobordism between objects c_1 and c_2, $\mathcal{F}(\Sigma_{c_1,c_2})$ is a linear operator

$$\mathcal{F}(\Sigma_{c_1,c_2}) : \mathcal{F}(c_1) \to \mathcal{F}(c_2).$$

Such a field theory must monoidal and respect gluing of surfaces:

- $\mathcal{F}(c_1 \sqcup c_2) \simeq \mathcal{F}(c_1) \otimes \mathcal{F}(c_2)$.
- Given a glued surface $\Sigma_1 \# \Sigma_2$, then

$$\mathcal{F}(\Sigma_1 \# \Sigma_2) = \mathcal{F}(\Sigma_1) \circ \mathcal{F}(\Sigma_2).$$



Examples

1. The string topology of a closed, oriented manifold M, $\mathcal{S}_M$ (Chas-Sullivan, Cohen-Jones, Cohen-Godin, Godin, Kupers)

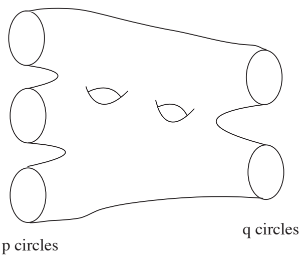
- $\mathcal{D} = \{N \subset M : N \text{ is closed, oriented, submanifold}\}$
- $\mathcal{S}_M(S^1) = H_*(LM)$.
- $\mathcal{S}_M(I_{N_2}^{N_1}) = H_*(P_M(N_1, N_2))$ where $P_M(N_1, N_2) = \{\gamma : [0,1] \to M, \text{ such that } \gamma(0) \in N_1,\ \gamma(1) \in N_2\}$.
- For a general one-manifold with labels, $\mathcal{S}_M(c) = H_*(Map(c, M; \partial))$.
- For a cobordism Σ from c_1 to c_2, consider the restrictions $Map(\Sigma, M; \partial)$ to the incoming and outgoing boundaries,

$$Map(c_1, M; \partial) \xleftarrow{\rho_{in}} Map(\Sigma, M; \partial) \xrightarrow{\rho_{out}} Map(c_2, M; \partial).$$

$$\mathcal{S}_M(\Sigma) : H_*(Map(c_1, M; \partial)) \xrightarrow{\rho_{in}^!} H_*(Map(\Sigma, M; \partial))$$
$$\xrightarrow{(\rho_{out})_*} H_*(Map(c_2, M; \partial)).$$

Defining $\rho_{in}^!$ rigorously involves intersection theory on spaces of paths and loops in M.

When $c_1 = p$ circles, $c_2 = q$ circles, and Σ is a cobordism.



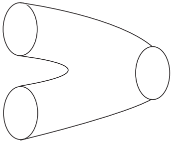
Yields operations

$$\mathcal{S}_M(\Sigma) : H_*(LM)^{\otimes p} \to H_*(LM)^{\otimes q}$$

More generally,

$$\mathcal{S}_M(\Sigma) : H_*(\mathcal{M}(\Sigma)) \otimes H_*(LM)^{\otimes p} \to H_*(LM)^{\otimes q}$$

$\mathcal{M}(\Sigma) =$ moduli space of curves diffeo to $\Sigma \simeq BDiff(\Sigma; \partial)$.
When Σ is the "pair of pants"



one gets the Chas-Sullivan closed string product,

$$H_q(LM) \otimes H_r(LM) \to H_{q+r-n}(LM).$$

Example 2. The Floer field theory of the cotangent bundle. T^*M.
Let $p : T^*M \to M$ be the cotangent bundle. Recall that T^*M has a *canonical* symplectic structure.
For $x \in M$, $u : T_xM \to \mathbb{R}$, define

$$\alpha(x, u) : T_{(x,u)}(T^*M) \xrightarrow{Dp} T_xM \xrightarrow{u} \mathbb{R}$$

$\alpha \in \Omega^1(T^*M)$ is the "Liouville 1-form".

$d\alpha = \omega \in \Omega^2(T^*M)$ is symplectic.

If $N \subset M$ is a submanifold, then its conormal bundle $cn(N) \subset T^*M$ is a Lagrangian submanifold. (A Lagrangian submanifold L of a symplectic manifold Q is defined by the property that $\omega(u, v) = 0$ for all $u, v \in T_xL$.)

Given an exact symplectic manifold (N^{2n}, ω) with $\omega = d\eta$, the Symplectic Floer homology, $SH_*(N, \omega)$. is defined by doing a type of infinite dimensional Morse theory on the free loop space, LN, using the symplectic action

$$\begin{aligned} \mathcal{A} : LN &\to \mathbb{R} \\ \gamma &\to \int_{S^1} \gamma^*(\eta) \end{aligned}$$

(Note if (N, ω) is not exact one can define $\mathcal{A}$ on the universal cover of LN.)

After perturbing if necessary, using a periodic time-dependent Hamiltonian, and choosing a compatible almost complex structure J, (which, together with the symplectic form defines a Riemannian metric) one gets a Morse-type chain complex (the "Floer complex")

$$\cdots \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \to \cdots$$

The boundary maps are defined by counting "J-pseudoholomorphic cylinders"

The resulting homology is $SH_*(N, \omega)$.

Now restrict to the case $(N, \omega) = (T^*M, \omega)$.

Theorem

(Viterbo, Abbondandolo-Schwarz, Salamon-Weber) If M is Spin, then

$$SH_*(T^*M, \omega) \cong H_*(LM).$$

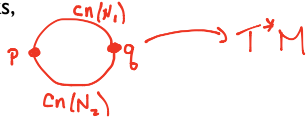
(If M is not spin, one must use twisted coefficients.)

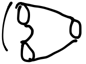
2). Floer symplectic field theory of T*M. $Symp_{T^*M}$

a. $Symp_{T^*M}(S^1) = SH_*(T^*M, \omega) \underset{\text{Viterbo}}{\simeq} H_* LM$

b. $Symp_{T^*M}\left(\overset{}{\underset{N_1 \quad N_2}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}}\right) = HF_*(T^*M;\, cn(N_1),\, cn(N_2))$

= "Lagrangian intersection Floer homology"
defined by a chain complex generated by intersection
points, $cn(N_1) \cap cn(N_2)$ (if transverse)
boundary homomorphisms defined by counting J-holomorphic
disks,



c. $Symp_{T^*M}$() $\quad SH_*(T^*M) \times SH_*(T^*M) \to SH_*(T^*M)$

Defined by counting J-holomorphic curves

Theorem

(C., Ganatra) Given any field k, there are 2D open-closed, positive boundary, topological field theories, $\mathcal{S}_M$ and $Symp_{T^*M}$ taking values in Chain Complexes over k, such that

1. *When one passes to homology they realize the above theories*
2. *There is a natural equivalence of chain complex valued field theories, $\Phi : Symp_{T^*M} \xrightarrow{\simeq} \mathcal{S}_M$.*

Idea:

Use recent methods of classifying TFT's:

- Cobordism hypothesis of Lurie
- Costello, Kontsevich-Vlassopolous

Roughly: $2D$ "positive boundary" oriented open-closed TFT's are classified by "Calabi-Yau (A)-∞ categories."

So we show: The string topology category $\mathcal{S}_M$ defined by Blumberg, C., Teleman is Calabi-Yau as is the "Wrapped Fukaya category" $\mathcal{W}(T^*M)$ defined by Seidel, Fukaya (this part was proved by Ganatra in his thesis) and that

$$\mathcal{S}_M \simeq \mathcal{W}(T^*M)$$

as CY A_∞-categories.

The notion of a Calabi-Yau category encodes the properties possessed by the category of coherent sheaves $Coh(X)$ on a Calabi-Yau variety X. In this case the corresponding field theory is the "B-model".

These categories are all enriched over chain complexes. Such a category with only one object is an DGA, so we describe these notions in this setting.

Let A be an (A_∞) algebra over a field k. Consider its Hochschild chains $CH_*(A) \simeq A \otimes^L_{A \otimes A^{op}} A$. It is an ($A_\infty$) module over $E(\Delta) \simeq C_*(S^1)$. The cyclic chains can be viewed as the homotopy orbits $CC_*(A) \simeq CH_*(A) \otimes^L_{E(\Delta)} k$.

Definition

(Kontsevich and coauthors, Costello, Lurie) Suppose that A is compact (perfect as a k-module). A compact Calabi-Yau (cCY) structure is a map

$$\bar{\tau} : CC_*(A) \to k$$

such that the composition

$\tau : A \otimes^L_{A \otimes A^{op}} A \simeq CH_*(A) \to CC_*(A) \xrightarrow{\bar{\tau}} k$ induces a pairing

$$A \otimes A \to k$$

that is homotopy nondegenerate in the sense that the adjoint $A \to A^*$ is an equivalence of A-bimodules. "self duality"

There is a related notion called a smooth Calabi category or sCY-category.

Given an A_∞-algebra or category A, let $CC_*^-(A)$ be the "negative cyclic chains". These chains can be viewed as the homotopy fixed points:

$$CC_*^-(A) \simeq Rhom_{E(\Delta)}(k, CH_*(A))$$

- An A_∞ algebra A is said to be "smooth" if is perfect as an A-bimodule. That is, it is perfect as a left module over $A \otimes A^{op}$.
- Let $A^!$ be the "bimodule dual" of A:

$$A^! = Rhom_{A \otimes A^{op}}(A, A \otimes A^{op})$$

Definition

A *sCY*-structure ("smooth Calabi-Yau") on a smooth A_∞-algebra A is an element

$$\bar{\sigma} \in CC_*^-(A)$$

So that if $\sigma \in CH_*(A)$ is the image under the natural map $CC_*^-(A) \to CH_*(A)$, then

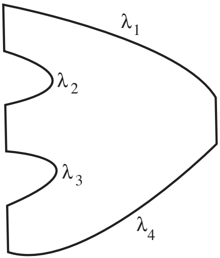
$$\cap\sigma : A^! \to A$$

$$Rhom_{A\otimes A^{op}}(A, A \otimes A^{op}) \to A \otimes^L_{A\otimes A^{op}} A \otimes A^{op} \simeq A \qquad (1)$$

is an equivalence of A-bimodules. "self duality as A-bimodules"

Theorem

(Kontsevich-Soibelman, Costello, Lurie) If $\mathcal{C}$ is a category with either a cCY category or a sCY structure, $\mathcal{C}$ gives rise to a positive boundary open-closed field theory $\mathcal{F}_\mathcal{C}$ with $\mathcal{F}_\mathcal{C}(S^1) \simeq CH_*(\mathcal{C})$. The boundary values ("D-branes") of the field theory are $\mathcal{D} = Ob\,\mathcal{C}$. The value of $\mathcal{F}_\mathcal{C}$ on the interval with endpoints labeled by λ_1, $\lambda_2 \in Ob\,\mathcal{C}$ is given by $Mor_\mathcal{C}(\lambda_1, \lambda_2)$. The value of $\mathcal{F}_\mathcal{C}$ on the open closed cobordism below is given by the higher composition laws in $\mathcal{C}$.



Theorem

(C. - Ganatra) The string topology category $\mathcal{S}_M$ and the wrapped Fukaya category $\mathcal{W}(T^*M)$ both have naturally occurring sCY-structures whose associated chain complex-valued field theories yield String topology and the Floer-symplectic field theories respectively (on the level of homology). Furthermore there is a natural equivalence $\mathcal{W}(T^*M) \xrightarrow{\simeq} \mathcal{S}_M$ that preserves these sCY-structures.

Note:

$\mathcal{S}_M$ is, roughly speaking, the category whose objects are closed, oriented, connected submanifolds $N \subset M$, and whose morphisms from N_1 to N_2 is equivalent to $C_*(P_M(N_1, N_2))$. Composition is equivalent to the open string product (Sullivan).

To make this rigorous, Blumberg, C., and Teleman constructed $\mathcal{S}_M$ as a full subcategory of the category of perfect modules over $C_*(\Omega M)$, generated by $C_*(P_M(pt, N))$. They proved that

$$\begin{aligned} Rhom_{C_*(\Omega M)}&(C_*(P_M(pt, N_1)), C_*(P_M(pt, N_2))) \\ &\simeq C_*(P_M(N_1, pt)) \otimes^L_{C_*(\Omega M)} C_*(P_M(pt, N_2)) \\ &\simeq C_*(P_M(N_1, N_2)) \end{aligned}$$

and that composition in these derived homomorphism spaces corresponds to the string product, defined using the Pontrjagin-Thom construction.

Since the endomorphisms of a point $End_{\mathcal{S}_M}(pt) = Rhom_{C_*(\Omega M)}(C_*(\Omega M), C_*(\Omega M)) \simeq C_*(\Omega M)$, then clearly $C_*(\Omega M)$ generates $\mathcal{S}_M$.

Abouzaid (2011) proved there is an equivalence of A_∞-algebras $End_{\mathcal{W}(T^*M)}(T^*_x M) \simeq C_*(\Omega M)$ and that $T^*_x M$ generates $\mathcal{W}(T^*M)$

Idea of proof Why is there a *sCY* structure on $\mathcal{S}_M$?

Lemma

If $\mathcal{C}_1 \subset \mathcal{C}_2$ generates (i.e the thick subcategory generated by $\mathcal{C}_1$ is $\mathcal{C}_2$), and if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are smooth, then $\mathcal{C}_1$ is sCY if and only if $\mathcal{C}_2$ is sCY.

Theorem

If M is a closed, oriented n-manifold, the $C_*(\Omega M)$ is sCY.

Note: $C_*(\Omega M) = End_{\mathcal{S}_M}(point)$. So by the lemma, this would prove that $\mathcal{S}_M$ is sCY.

Sketch of proof. Recall Goodwillie proved that

$$CH_*(C_*(\Omega M)) \simeq C_*(LM).$$

Also observe

$$LM^{hS^1} = Map_{S^1}(ES^1, LM) = Map_{S^1}(ES^1 \times S^1, M) \simeq M.$$

So therefore there is a chain map

$$C_*(M) \simeq C_*(LM^{hS^1}) \to Rhom_{C_*(S^1)}(k, CH_*(C_*(\Omega M)) \quad (2)$$
$$= CC_*^-(C_*(\Omega M)). \quad (3)$$

Definition

We say that class $\bar{\sigma} \in CC_*^-(C_*(\Omega M))$ is of *fundamental type* if its homology class $[\bar{\sigma}] \in HC^-(C_*(\Omega M))$ is the image of the fundamental class

$$H_*(M) \to HC_*^-(C_*(\Omega M)) \quad (4)$$
$$[M] \to [\bar{\sigma}]. \quad (5)$$

Claim. Any class $\bar{\sigma} \in CC_*^-(C_*(\Omega M))$ of fundamental type defines a sCY structure on $C_*(\Omega M)$.

Proof. Let $A = C_*(\Omega M)$. We need to show that if $\sigma \in CH_*(A)$ is the image of $\bar{\sigma} \in CC_*^-(A)$, then

$$\cap\sigma : Rhom_{A \otimes A^{op}}(A, A \otimes A^{op}) \to A$$

is an equivalence.

That is, we need to show

$$\cap[\sigma] : Ext_{A \otimes A^{op}}(A, P) \to Tor_{A \otimes A^{op}}(A, P)$$

is an isomorphism, where $P = A \otimes A^{op}$.
Now since $A = C_*(\Omega M)$ is a connective Hopf algebra, $Ext_{A \otimes A^{op}}(A, P) \cong Ext_A(k, P^{ad})$. (Similarly for Tor).

Since $A = C_*(\Omega M)$ this becomes

$$\cap[\sigma] : H^*(M; P^{ad}) = Ext_{C_*(\Omega M)}(k, P^{ad}) \to Tor_{C_*(\Omega M)}(k, P^{ad}) \\ = H_*(M, P^{ad})$$

(coefficients are twisted by modules over $C_*(\Omega M)$.)

Since $\bar{\sigma}$ is of fundamental type, the fact that this is an isomorphism is Poincaré duality with these twisted coefficients (Dwyer-Greenlees-Iyengar).

Ganatra proved that $\mathcal{W}(T^*M)$ is sCY in his thesis. Moreover we have a functor defined by a variant of a construction of Abbondandolo and Schwarz,

$$AS : \mathcal{W}(T^*M) \to \mathcal{S}_M$$

which is seen to be an equivalence of categories by an argument of Abouzaid. Now must check that the sCY-structures are preserved. (Technically the most complicated.)

There are two other features.

1. We say that an augmented DGA A is "strongly smooth" if A is smooth and k is a perfect module over A (so in particular $Tor_A(k,k)$ is finite.) $C_*(\Omega M)$ is strongly smooth if M is closed.

Theorem

Let A be a strongly smooth DGA over k. Suppose B is a DGA that is Koszul dual to A. That is,

$$B \simeq Rhom_A(k,k) \quad A \simeq Rhom_B(k,k).$$

They A is sCY if and only if B is cCY. Furthermore, their associated field theories $\mathcal{F}_A$ and $\mathcal{F}_B$ are dual.

Note: Since A and B are Koszul dual, $HH_*(A) \cong HH_*(B)^*$ (Jones-McCleary) (For THH this is due to J. Campbell.)

Example $A = C_*(\Omega M)$, $B = C^*M$, M simply connected.

Lurie's cobordism hypothesis says that an extended TFT with values in $\mathcal{C}$ (a symmetric monoidal $(\infty, 2)$-category) are classified by "Calabi-Yau objects" in $\mathcal{C}$.

Conjecture 1. A is a cCY category in the sense of Kontsevich if and only if A is a CY object in the sense of Lurie in the $(\infty, 2)$-category $\mathcal{C}AT =$ Categories, Functors, and Natural Transformations.

2. A is a sCY category in the sense of Kontsevich if and only if A is a CY object in the sense of Lurie in $\mathcal{C}AT^{op}$.

Caution: Need finiteness conditions!

This is a joint project with Ganatra and A. Blumberg.