Kernel Methods for Persistence Diagrams

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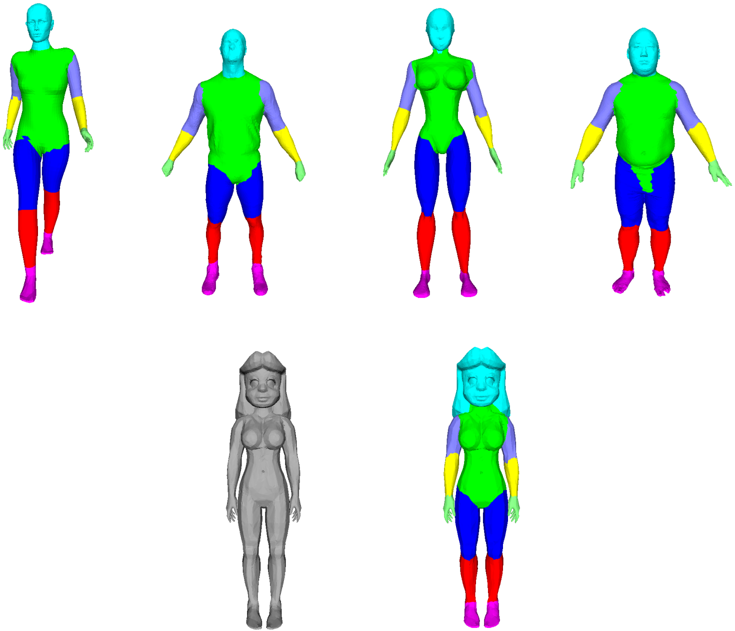
Introduction



Statistical Learning

- Characterize shape points $x \in S$ with **signature vectors** $X \in \mathbb{R}^d$
- Assume $(X, Y) \sim P$
- Find a classifier $f : X \mapsto Y = f(X)$

Statistical Learning

- Assume shape S has two labels (segments) $Y = -1/+1$
- Select classifier that minimizes the following loss:

$$\mathbb{E}_P[\phi(Yf(X))] + \|f\|$$

- Support Vector Machine (SVM) use the *hinge* loss: $\phi(u) = \max(0, 1-u)$
- Linear SVM: $f(X) = a^T X + b$
- Kernel SVM: $f(X) = a^T \Psi(X) + b$ where Ψ is a mapping to a RKHS
- When more than 2 labels: multi-class SVM

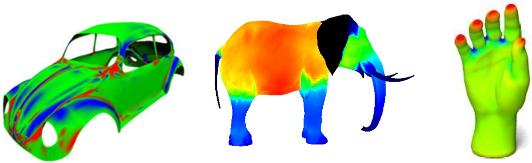
Introduction

- Shape = triangle mesh
- "Good" signatures are necessary for a good model
- **Can be very different by nature:**
  - local/global
  - intrinsic/extrinsic
  - volumetric/defined on the surface
  - type of information (geometry, topology...)
- **Satisfy the following properties:**
  - be *invariant* to deformation classes (rotation, scaling...)
  - be *stable*
  - be *informative*
  - bring *complementary* information to common signatures
  - be representable as vectors in $\mathbb{R}^d$

Introduction

Examples:

- curvature (mean, gaussian)
- PCA features
- spin image
- shape context
- shape diameter function
- kernel signatures (heat kernel, wave kernel)
- geodesic features (eccentricity)



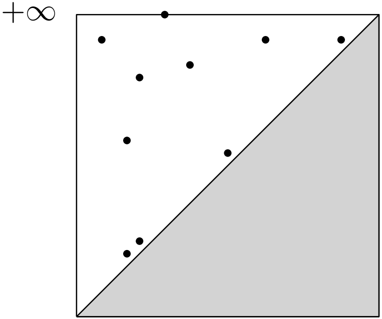
Introduction

- **General context:** use *persistent homology* to build topological signatures
- **Issues with existing techniques:**
  - global
  - costly to compute
  - not well suited for learning
- **Contribution:** local topological efficient and provably stable signature in $\mathbb{R}^d$

Persistence Diagrams

Persistence Diagrams

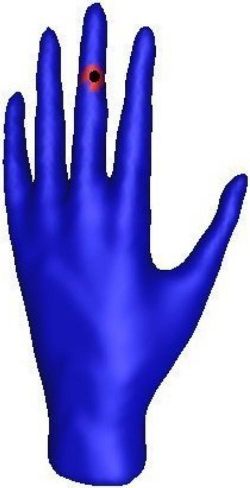
- Persistence Diagrams (PDs) are the building blocks of the topological signature
- PDs are sets of points in $\bar{\mathbb{R}}^2$

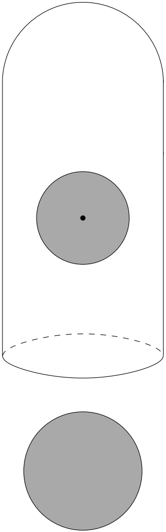


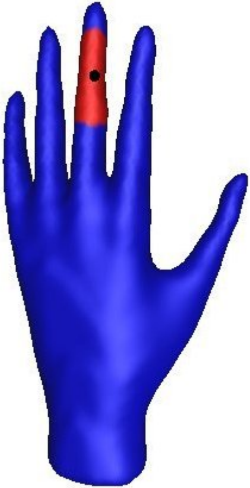
Persistence Diagrams

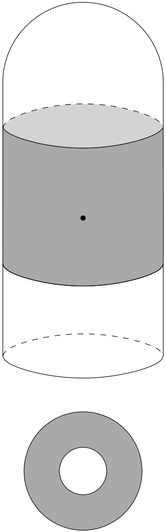
Pick a point x

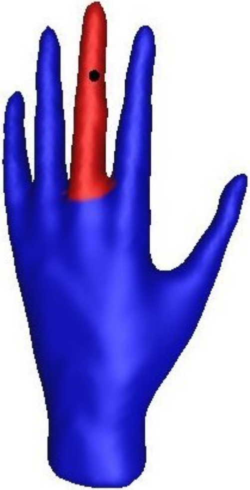
- Record topological changes (i.e. *homology*) of growing geodesic ball centered on x
- $\rightarrow$ Record appearance and filling of every hole in the ball
- For every hole, create point (x, y) in PD with
  - $x =$ radius of appearance
  - $y =$ radius of filling

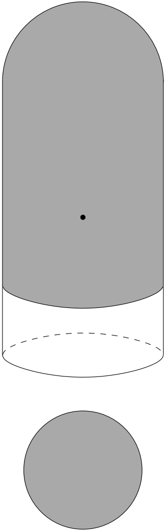


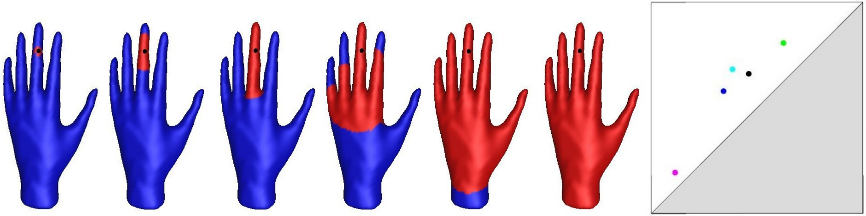


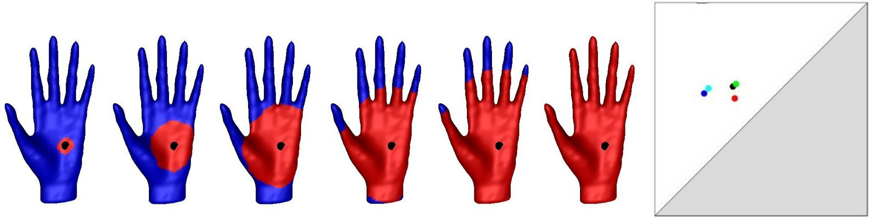












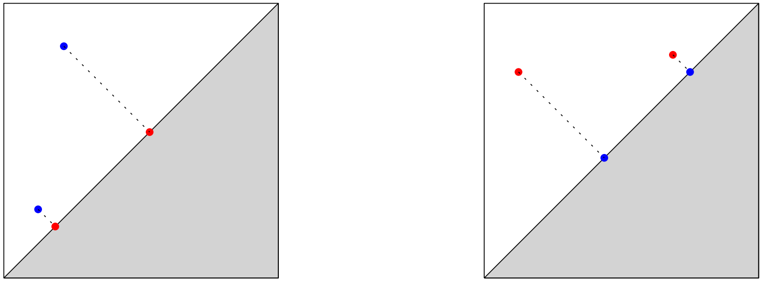
- Stability?
- Distance between PDs?
- Distance between shapes?

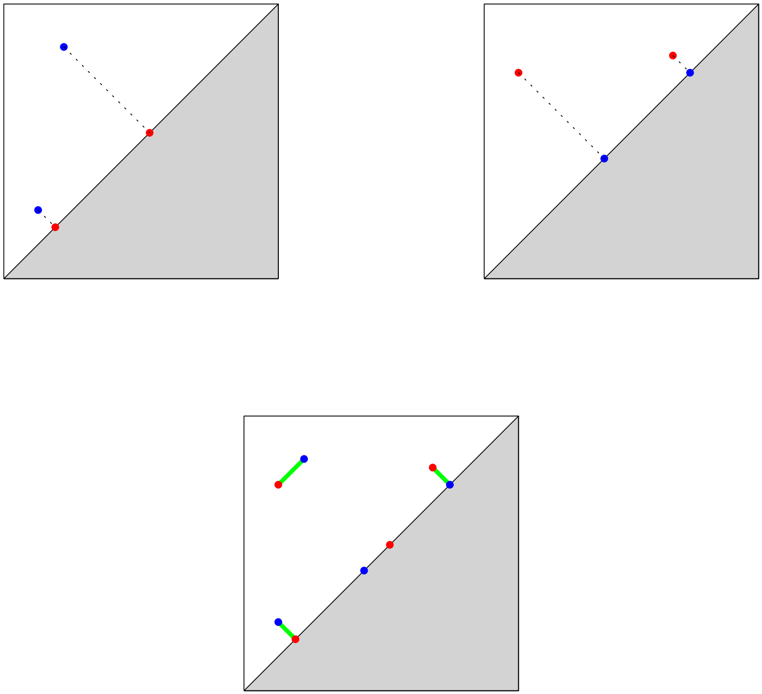
Distance between PDs

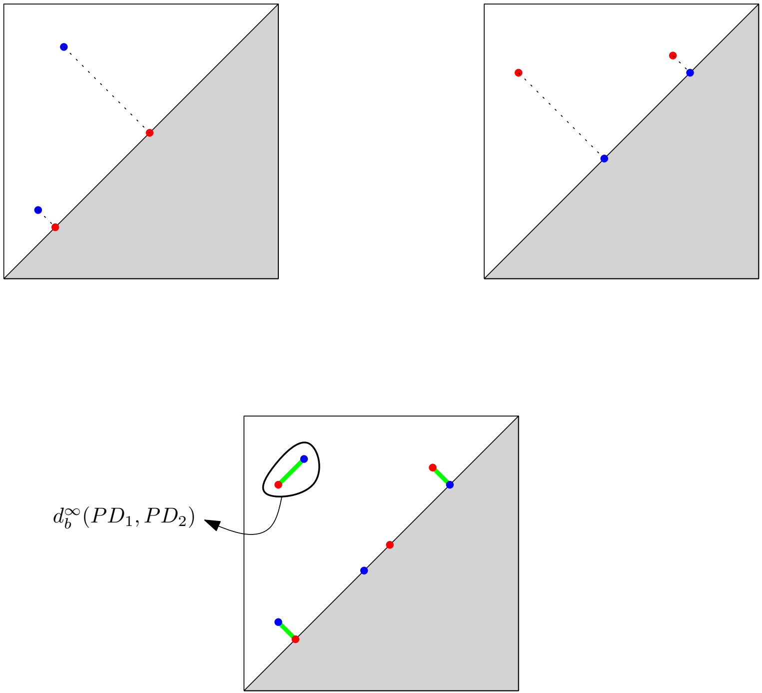
Let $\mathrm{PD} = \{p_1 \ldots p_n\}$ and $\mathrm{PD}' = \{q_1 \ldots q_m\}$ be two PDs.
Let $S = \mathrm{PD} \cup P_\Delta(\mathrm{PD}')$ and $S' = \mathrm{PD}' \cup P_\Delta(\mathrm{PD})$ ($|S| = |S'|$).
Then:

$$d_b^\infty(\mathrm{PD}, \mathrm{PD}') = \inf_{\phi: S \to S'} \sup_{i=1 \ldots n} c(p_i, \phi(p_i))$$

where $c(p_i, \phi(p_i)) = \|p_i - \phi(p_i)\|_\infty$





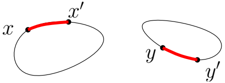

$d_b^{\infty}(PD_1, PD_2)$

Distance between shapes

- A *correspondence* between metric spaces X and Y is a subset C of $X \times Y$ such that:
  - $\forall x \in X, \exists y \in Y$ s.t. $(x, y) \in C$
  - $\forall y \in Y, \exists x \in X$ s.t. $(x, y) \in C$

- The *metric distortion* $\epsilon_m(C)$ of C is:

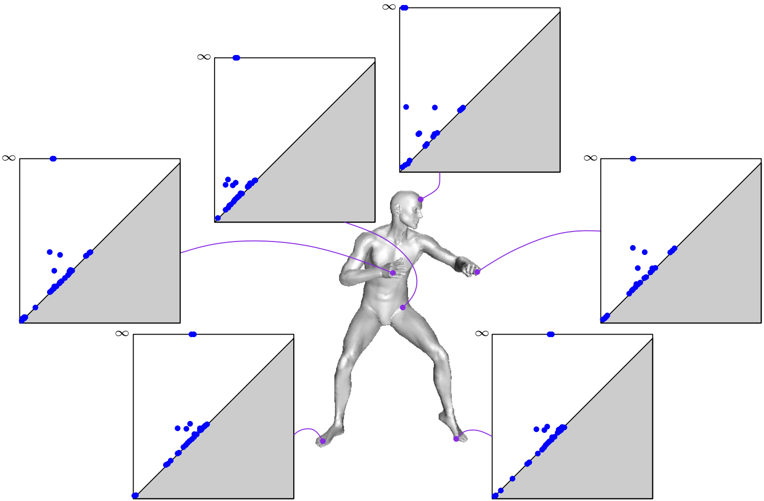
$$\epsilon_m(C) = \sup_{(x,y)\in C,(x',y')\in C} |d_X(x,x') - d_Y(y,y')|$$

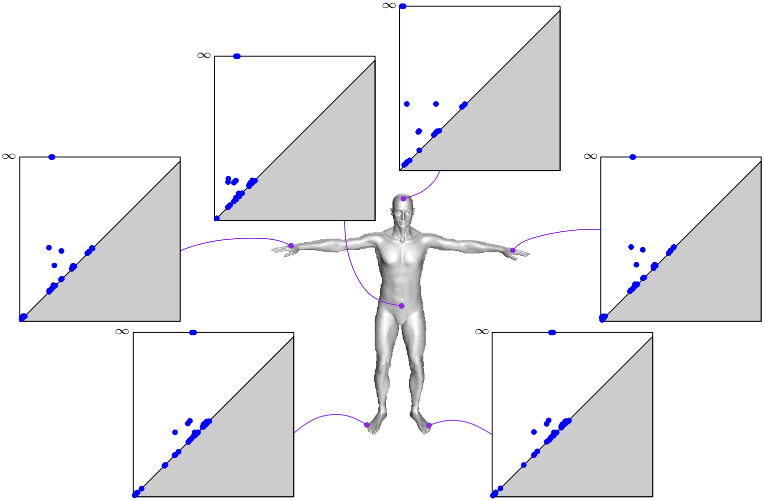


Stability

Theorem: Corresponding points in nearly-isometric shapes have similar PDs:

$$d_b^\infty(D_x, D_y) \leq 20 \inf_{C:(x,y)\in C} \epsilon_m(C)$$





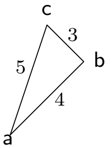
Kernels

Kernel

- d_b^∞ = cost of optimal matching → costly to compute in practice
- Kernel SVM? $K(D, D') = \exp\left(-\frac{d_b^\infty(D,D')^2}{2\sigma^2}\right)$
- d_b^∞ is not conditionally negative definite → K is not a valid kernel
- **Idea:** see PDs as metric spaces to turn them into vectors

Feature Map

finite metric space



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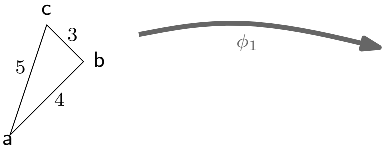
Feature Map

finite metric space

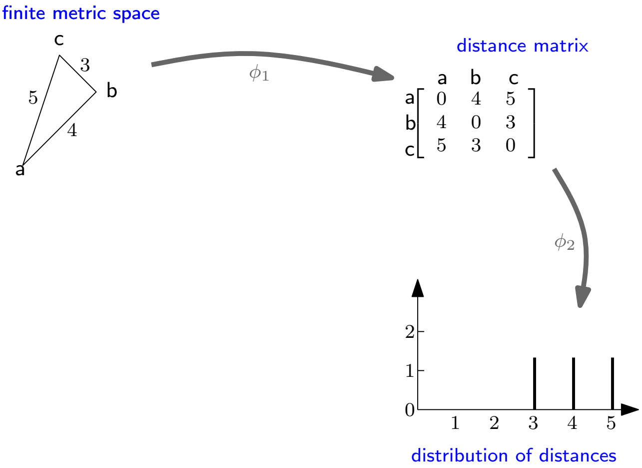


distance matrix

	a	b	c
a	0	4	5
b	4	0	3
c	5	3	0

3

Feature Map



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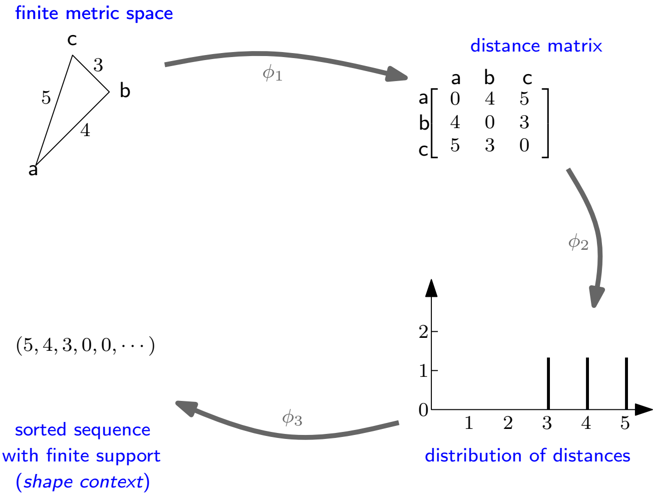

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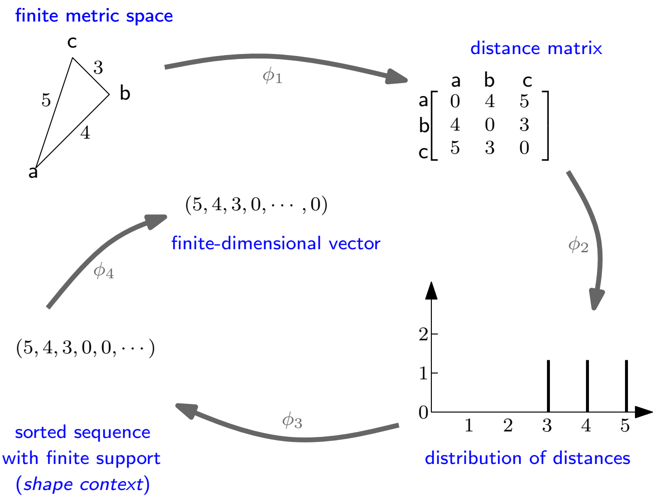
Feature Map

$\Phi = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$



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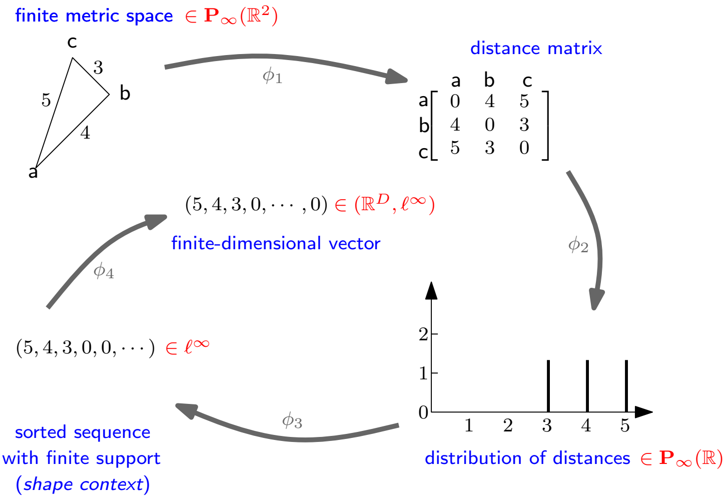

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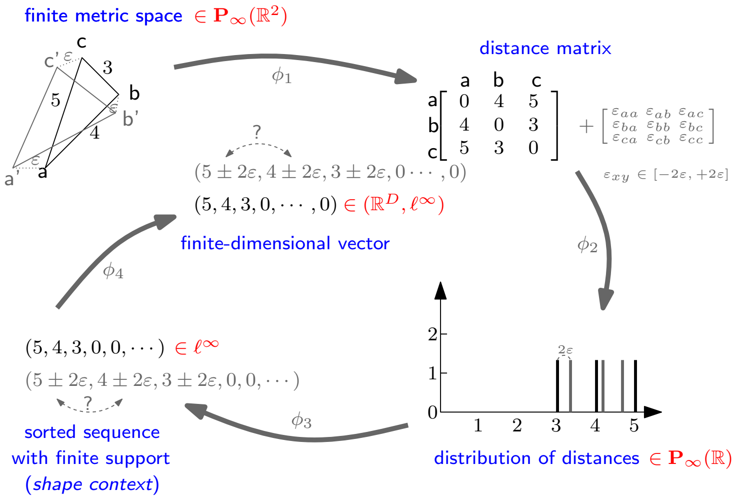
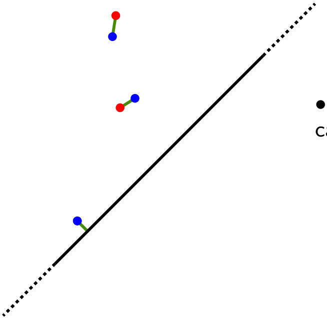


3

Adding the diagonal

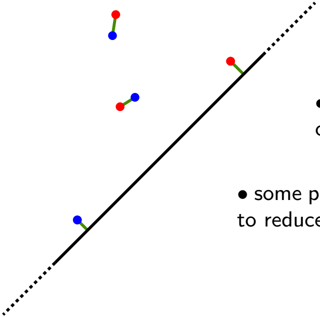


• diagonal has infinite multiplicity

• useful for when point clouds have different cardinalities

4

Adding the diagonal



• diagonal has infinite multiplicity

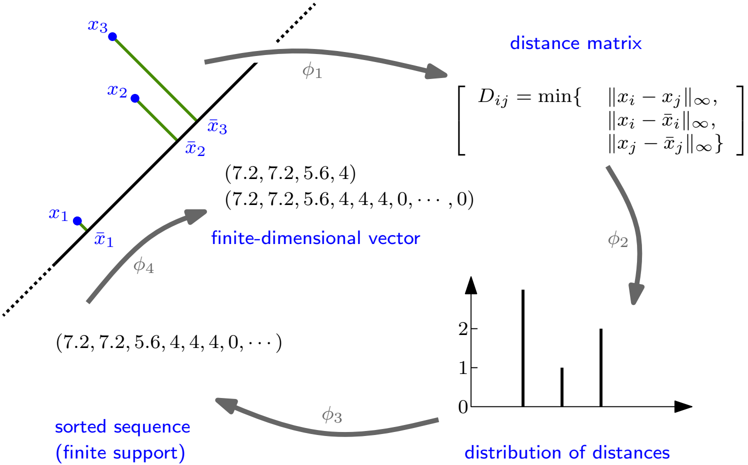
• useful for when point clouds have different cardinalities

• some points may prefer the diagonal to other points to reduce the cost of the matching

4

Adding the diagonal

$\Phi = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$



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Stability

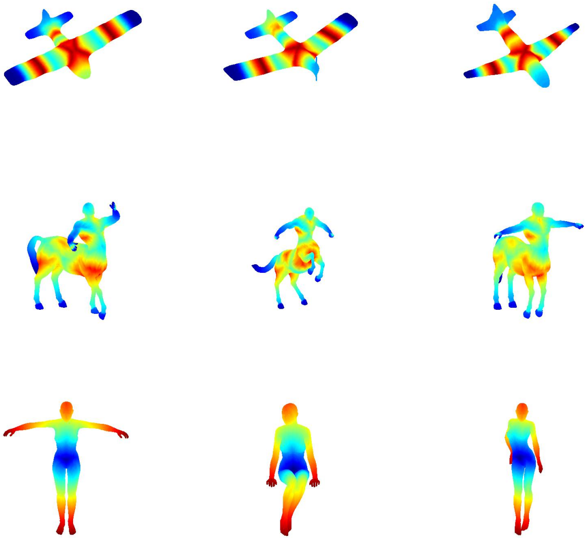
$$K_1(D_x, D_y) =< \Phi(D_x), \Phi(D_y) >$$

$$K_2(D_x, D_y) = \exp\left(-\frac{\|\Phi(D_x) - \Phi(D_y)\|_2^2}{2\sigma^2}\right)$$

$$C(N)\|\Phi(D_x) - \Phi(D_y)\|_2 \leq \|\Phi(D_x) - \Phi(D_y)\|_\infty \leq 2d_b^\infty(D_x, D_y)$$

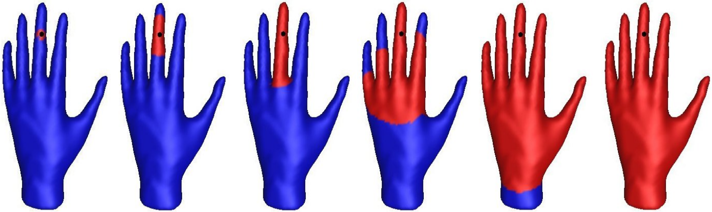
- $C(N) = \sqrt{\frac{2}{N(N-1)}}$ where N is the dimension
- Stability preserved **whatever the number of components kept!**

Stability



Computation

- Symmetry: count *connected components* instead of holes
- CCs are computed with triangulation + Dijkstra's algorithm
- Can be extended to point clouds with neighborhood graph

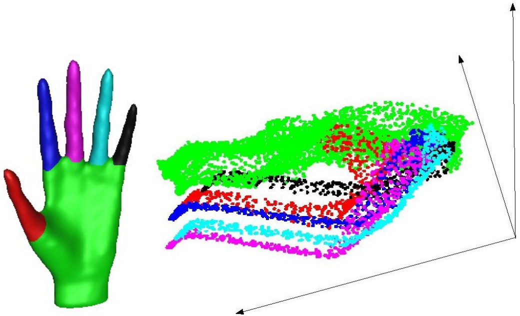


Computation

- Use Union Find data structure
- Timing:
  - 3-5 min for shape with 10k-15k nodes
  - 15 min for shape with 30k nodes
  - Computation of distance matrix $\simeq 66\%$
  - Computation of PDs $\simeq 33\%$
- Complexity:
  - Distance Matrix: $O(n^2\log(n))$
  - PDs: $O(n^2\log(n))$
  - Mapping: $O(n^3)$ – in practice $O(n)$
- Code available at http://geometrica.saclay.inria.fr/team/Mathieu.Carriere/

Continuity

- Kernel PCA with K_1



- Values vary smoothly over the shape

Not the only way to derive kernels...

- Heat diffusion map $L^2(\mathbb{R}^2)$, stability with Wasserstein distance
- Landscapes $L^2(\mathbb{R}^2)$
- Roots of complex polynomials $\mathbb{R}^d$

Application 1: Shape Segmentation

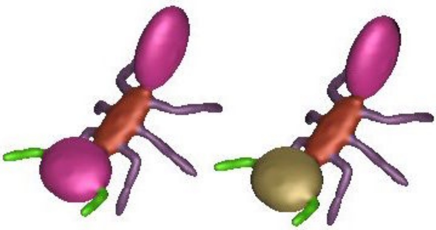
Shape Segmentation

- Learning on training set with/without topological signatures
- Smoothing of produced segmentation (graphcut algorithm)
- Evaluate segmentation with Rand Index

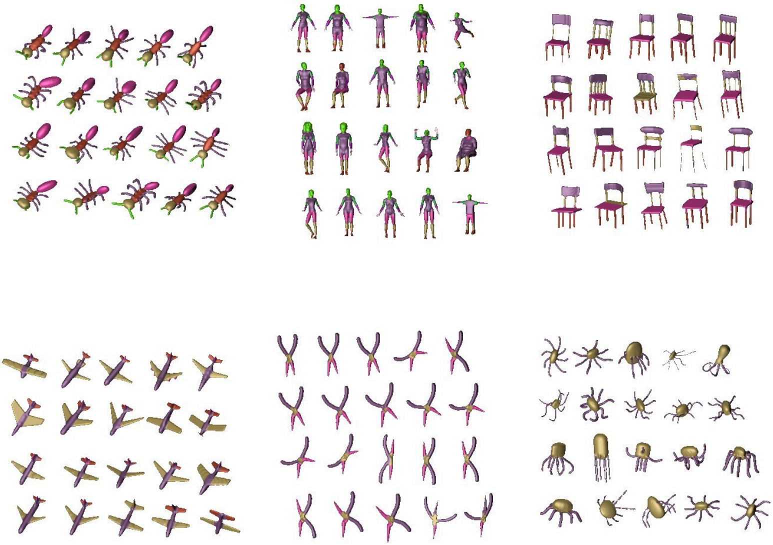
Results

	SB5	SB5+PDs
Human	21.3	**11.3**
Cup	10.6	**10.1**
Airplane	18.7	**9.3**
Ant	9.7	**1.5**
Chair	15.1	**7.3**
Octopus	5.5	**3.4**
Table	7.4	**2.5**
Teddy	6.0	**3.5**
Hand	21.1	**12.0**
Plier	12.3	**9.2**
Fish	20.9	**7.7**
Bird	24.8	**13.5**

Results



Results



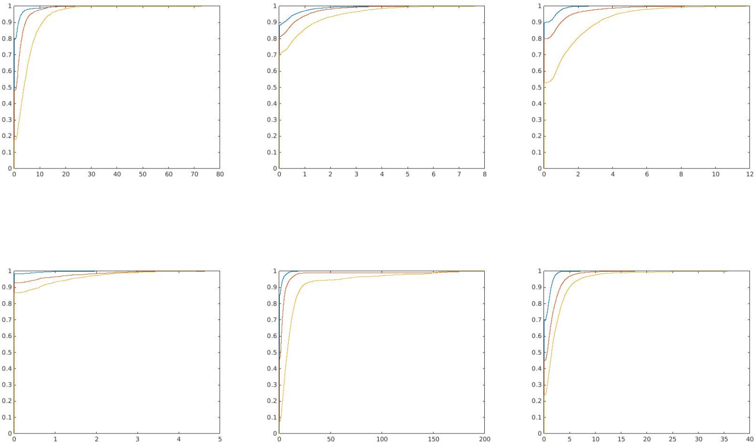
Application 2: Shape Matching

Shape Matching

- Compute optimal map $S_1 \to S_2$ that best preserves a set of signatures (with/without topological signature)
- Derive correspondence from this map
- Evaluate quality of correspondence

Results

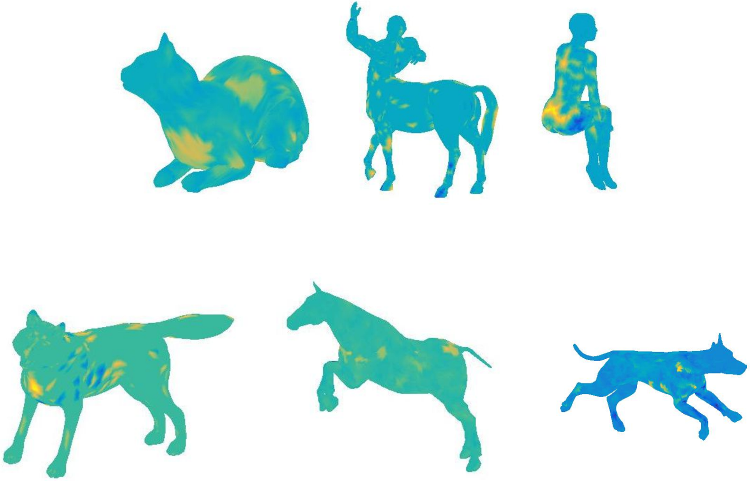
Percentage y of points that are mapped at distance at most x from their ground truth images (equivalent of Precision-Recall curve)



Results

Flat regions are improved

Results



Conclusion

- We introduced provably stable topological multiscale signature and kernel for points in shapes that gives complementary information to the other classical signatures
- Drawbacks: not well suited for all shapes, mapping loses information

Thank you!