# Voronoi diagram on a Riemannian surface

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# Motivation

**Aim** : Show a link between mean characteristics of the Voronoi cells and local characteristics of the surface

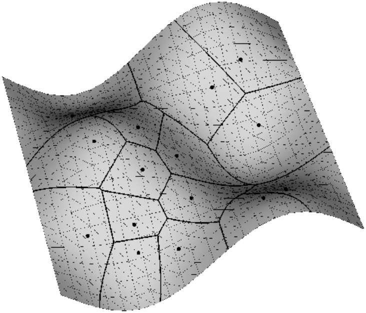


image:R.Kunze

# Framework

- $S$ Riemannian surface, with its Riemannian metric $d$,
- $dx$ area measure induced by the metric,
- $\Phi$ Poisson point process of intensity $\lambda dx$ and $x_0 \in S$ added to $\Phi$,
- The Voronoi cell of $x_0$ defined by

$$C(x_0, \Phi) = \{y \in S, d(x_0, y) \leq d(x, y), \forall x \in \Phi\}$$

- $N$ the number of vertices.

# Outline

1. Case of the sphere
2. Arbitrary surface
3. Ongoing work on the dimension $\geq 3$

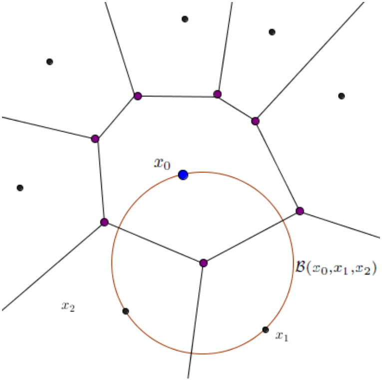
# Mean number of vertices

wlog, assume $x_0$ to be the North pole on the sphere of constant curvature $K$ (of radius $\frac{1}{\sqrt{K}}$)

$$\mathbb{E}[N(\mathcal{C})] = 6 - \frac{3K}{\pi\lambda} + e^{-\frac{4\pi\lambda}{K}}\left(\frac{3K}{\pi\lambda} + 6\right)$$

Miles (1971) : $n$ uniform points on the sphere

# Sketch of proof




**Step 1:** characterize vertices of $\mathcal{C}$

# Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \mathbb{E}\left[\sum_{x_1, x_2 \in \Phi} \mathbb{1}_{\{\mathcal{B}_1(x_0,x_1,x_2)\cap\Phi=\emptyset\}} + \mathbb{1}_{\{\mathcal{B}_2(x_0,x_1,x_2)\cap\Phi=\emptyset\}}\right]$$

**Step 1:** characterize vertices of $\mathcal{C}$

# Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \frac{\lambda^2}{2} \iint_{x_1, x_2 \in \mathcal{S}(K)} \left( e^{-\lambda \operatorname{vol}(\mathcal{B}_1(x_0, x_1, x_2))} + e^{-\lambda \operatorname{vol}(\mathcal{B}_2(x_0, x_1, x_2))} \right) dx_1 dx_2$$

**Step 2:** apply Mecke-Slivnyak formula

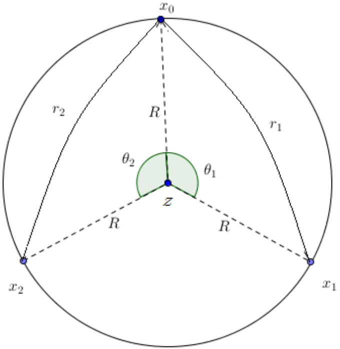
# Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \frac{\lambda^2}{2} \int_{r_1,\varphi_1,r_2,\varphi_2} \left( e^{-\lambda \operatorname{vol}(\mathcal{B}_1(x_0,x_1,x_2))} + e^{-\lambda \operatorname{vol}(\mathcal{B}_2(x_0,x_1,x_2))} \right)$$

$$\times \frac{\sin(\sqrt{K} r_1)}{\sqrt{K}} \frac{\sin(\sqrt{K} r_2)}{\sqrt{K}} dr_1 d\varphi_1 dr_2 d\varphi_2$$

**Step 3:** use spherical coordinates

# Sketch of proof




$$r_1 = \frac{2}{\sqrt{K}} \arcsin\left(\sin\left(\frac{\theta_1}{2}\right)\sin(\sqrt{K}R)\right)$$

$$r_2 = \frac{2}{\sqrt{K}} \arcsin\left(\sin\left(\frac{\theta_2}{2}\right)\sin(\sqrt{K}R)\right)$$

$$\varphi_1 = \varphi + \frac{\pi}{2} - \arctan\left(\tan\left(\frac{\theta_1}{2}\right)\cos(\sqrt{K}R)\right)$$

$$\varphi_2 = \varphi + \frac{\pi}{2} - \arctan\left(\tan\left(\frac{\theta_2}{2}\right)\cos(\sqrt{K}R)\right)$$

**Step 4:** make a Blaschke-Petkantschin type change of variables

# Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = 4\pi\lambda^2 I \int_0^{\frac{\pi}{2\sqrt{K}}} \left( e^{-\lambda \frac{2\pi}{K}(1-\cos(\sqrt{K}R))} + e^{-\lambda \frac{2\pi}{K}(1+\cos(\sqrt{K}R))} \right) \frac{\sin^3(\sqrt{K}R)}{\sqrt{K}} dR$$
$$= 6 - \frac{3K}{\pi\lambda} + e^{-\frac{4\lambda\pi}{K}} \left( 6 + \frac{3K}{\lambda\pi} \right)$$

where

$$I = \int_{\theta_1, \theta_2 \in [0, 2\pi]} \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) \left| \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \right| d\theta_1 d\theta_2$$

**Step 4:** Make a Blaschke-Petkantschin type change of variables

# Strategy

Find a way to adapt the method to a general surface

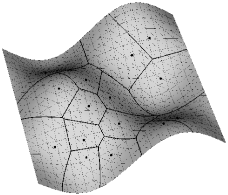


image:R.Kunze

- **Step 1:** characterize vertices of $\mathcal{C}$
- **Step 2:** apply Mecke-Slivnyak formula
- **Step 3:** use geodesic polar coordinates
- **Step 4:** make a Blaschke-Petkantschin type change of variables
- **Step 5:** find the volume of a geodesic ball

# Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \mathbb{E}\left[\sum_{x_1,x_2\in\Phi}\ \sum_{\text{circumscribed balls}} \mathbb{1}_{\{\mathcal{B}(x_0,x_1,x_2)\cap\Phi=\emptyset\}}\right]$$

**Step 1:** characterize vertices of $\mathcal{C}$

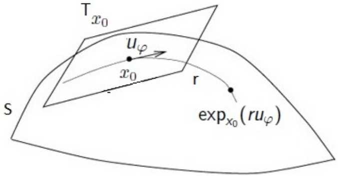
# Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \frac{\lambda^2}{2} \iint_{x_1, x_2 \in S} \sum_{\text{circumscribed balls}} e^{-\lambda \operatorname{vol}(\mathcal{B}(x_0, x_1, x_2))} dx_1 dx_2$$

1. Points "far" from $x_0$ contribute negligibly.
2. For points around $x_0$, we need similar changes of variables.

**Step 2:** apply Mecke Slivnyak formula

# Exponential map




Around $x_0$, $S$ can always be parametrized by its geodesic polar coordinates $(r, \varphi)$, ie

$$x = \exp_{x_0}(ru_\varphi)$$

**Step 3:** use geodesic polar coordinates

# Rauch theorem

$$dx = f(r,\varphi)drd\varphi$$

Let $K$ denote the Gaussian curvature.

**Rauch theorem (1951)**

Si $0 < \delta \leq K \leq \Delta$

$$\frac{\sin(\sqrt{\Delta}r)}{\sqrt{\Delta}} \leq f(r,\varphi) \leq \frac{\sin(\sqrt{\delta}r)}{\sqrt{\delta}}$$

*Application:* $\delta = K(x_0) - \varepsilon$, $\Delta = K(x_0) + \varepsilon$

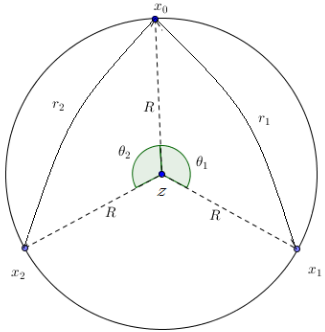
**Step 3:** use geodesic polar coordinates

# Sketch of proof

$$E[N(\mathcal{C})] = \frac{\lambda^2}{2} \int_{\substack{(r_1,\varphi_1)\\(r_2,\varphi_2)}} e^{-\lambda \operatorname{vol}(\mathcal{B}(x_0,x_1,x_2))}$$
$$\times \left(r_1 - \tfrac{K(x_0)r_1^3}{6} + o(r_1^3)\right)\left(r_2 - \tfrac{K(x_0)r_2^3}{6} + o(r_2^3)\right) dr_1 d\varphi_1 dr_2 d\varphi_2 + O\left(e^{-c\lambda}\right)$$

**Step 3:** use geodesic polar coordinates

# Sketch of proof




$r_1 =?$

$r_2 =?$

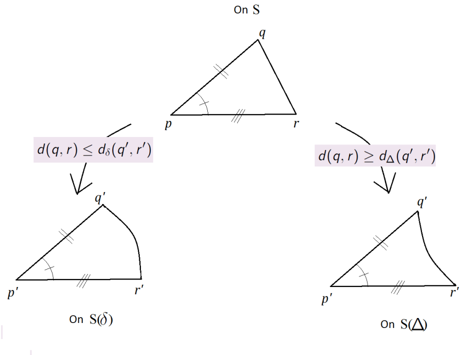
$\varphi_1 =?$

$\varphi_2 =?$

**Step 4:** make a Blaschke-Petkantschin type change of variables

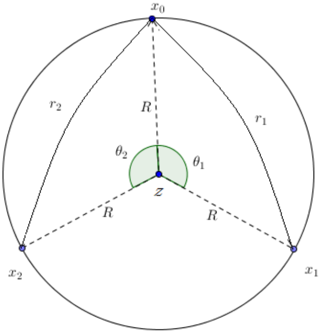
# Toponogov theorem

If $\delta \leq K \leq \Delta$




**Step 4:** make a Blaschke-Petkantschin type change of variables

# Sketch of proof




$$r_1 = 2\sin(\theta_1/2)R - \frac{K(x_0)R^3}{3}\sin(\theta_1/2)\cos^2(\theta_1/2) + o(R^3)$$

$$r_2 = 2\sin(\theta_2/2)R - \frac{K(x_0)R^3}{3}\sin(\theta_2/2)\cos^2(\theta_2/2) + o(R^3)$$

$$\varphi_1 = \varphi + \frac{\pi}{2} - \frac{\theta_1}{2} + \frac{K(x_0)R^2}{4}\sin(\theta_1) + o(R^2)$$

$$\varphi_2 = \varphi + \frac{\pi}{2} - \frac{\theta_2}{2} + \frac{K(x_0)R^2}{4}\sin(\theta_2) + o(R^2)$$

**Step 4:** make a Blaschke-Petkantschin type change of variables

# Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = 2\lambda^2 I \int_{\varphi} \int_{R} e^{-\lambda \operatorname{vol}(\mathcal{B}(z,R))} \left( R^3 - \tfrac{K(x_0)R^5}{2} + o(R^5) \right) dR d\varphi + O(e^{-c\lambda})$$

where

$$I = \int_{\theta_1,\theta_2} \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) \left| \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \right| d\theta_1 d\theta_2$$

**Step 4:** make a Blaschke-Petkantschin type change of variables

# Volume of small geodesic balls

**Bertrand-Diquet-Puiseux theorem (1848)**

When $r \to 0$, $x \in S$

$$\mathrm{vol}(\mathcal{B}(z, r)) = \pi r^2 - \frac{K(z)\pi}{12} r^4 + o(r^4)$$

**Step 5:** find the volume of the circumscribed ball

# Result

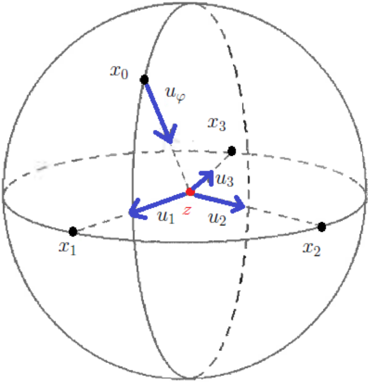
$$\mathbb{E}[N(\mathcal{C})] = 12\pi^2\lambda^2 \int_0^{R_{max}} e^{-\lambda(\pi R^2 - \frac{\pi K(x_0)R^4}{12} + o(R^4))} \times [R^3 - \frac{K(x_0)R^5}{2} + o(R^5)]dR + O(e^{-c\lambda})$$

When $\lambda$ goes to infinity, Laplace's method yields

## Mean number of vertices

$$\mathbb{E}[N(\mathcal{C})] = 6 - \frac{3K(x_0)}{\pi\lambda} + o\left(\frac{1}{\lambda}\right)$$

# The dimension $n$




The vertices of $\mathcal{C}$

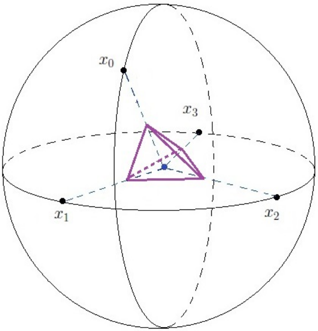
# The $n$-sphere of constant sectionnal curvature $K$

The Jacobian of the Blaschke-Petkantschin type change of variables (Miles 1971):

$$J = n! \left( \frac{\sin(\sqrt{K}R)}{\sqrt{K}} \right)^{n^2-1} \Delta(x_0, x_1, \ldots, x_n)$$

The volume of a ball of radius $R$ in $\mathcal{S}^n(K)$:

$$V(R) = \frac{2\pi^{\frac{1}{2}}}{\Gamma(\frac{n}{2})} \int_0^R \sin^{n-1}(t)dt$$




$\Delta(x_0, x_1, \ldots, x_n)$

# The $n$-sphere of constant sectionnal curvature $K$

## Mean number of vertices

$$\mathbb{E}[N(\mathcal{C})] = E_n - \frac{\mathrm{Sc}}{\lambda^{\frac{2}{n}}} C_n + o\left(\frac{1}{\lambda^{\frac{2}{n}}}\right)$$

where

- $E_n$ is the mean number of vertices in $\mathbb{R}^n$
- $C_n$ is a positive constant
- $\mathrm{Sc} = n(n-1)K$ is the scalar curvature of $\mathcal{S}^n(K)$

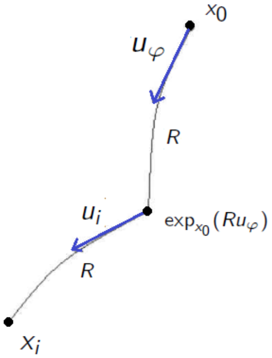
$$E_n = 2\pi^{\frac{n-1}{2}} n^{n-2} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\right)^n \frac{\Gamma(\frac{n^2+1}{2})}{\Gamma(\frac{n^2}{2})}$$

$$C_n = \frac{2^{1-\frac{2}{n}}}{6n!} \pi^{\frac{n}{2}-\frac{3}{2}} \frac{n^3-2}{(n-1)(n+2)} n^{n+\frac{2}{n}-2} \frac{\Gamma(n+\frac{2}{n})\Gamma(\frac{n}{2})^{n+\frac{2}{n}}\Gamma(\frac{n^2+1}{2})}{\Gamma(\frac{n^2}{2})\Gamma(\frac{n+1}{2})^n}$$

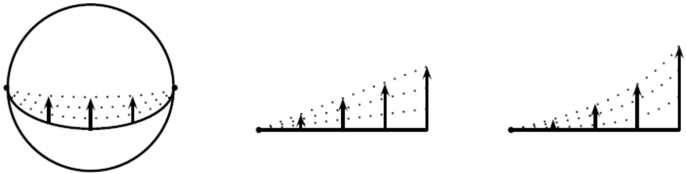
# Generalization to a $n$-manifold $M$

The Blaschke Petkantschin type change of variables is written as

$$x_i = \exp_{|\exp_{x_0}(Ru_\varphi)}(Ru_i)$$



- The Jacobian of this change of variables involves Jacobi fields: Rauch Theorem



Jacobi fields

- An expansion of the volume of a small geodesic ball on $M$ is given by

$$\mathrm{vol}(\mathcal{B}(z,R)) = \kappa_n R^n \left(1 - \frac{\mathrm{Sc}(z)}{6(n+2)} R^2 + o(R^2)\right)$$

# Generalization to a $n$-manifold $M$

## Conjecture for the mean number of vertices

$$\mathbb{E}[N(\mathcal{C})] = E_n - \frac{\mathrm{Sc}(x_0)}{\lambda^{\frac{2}{n}}} C_n + o(\frac{1}{\lambda^{\frac{2}{n}}})$$

with

- $E_n$ and $C_n$ the same constants as for the sphere
- $\mathrm{Sc}(x_0)$ is the scalar curvature of $M$ at $x_0$

$$E_n = 2\pi^{\frac{n-1}{2}} n^{n-2} \left( \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \right)^n \frac{\Gamma(\frac{n^2+1}{2})}{\Gamma(\frac{n^2}{2})}$$

$$C_n = \frac{2^{1-\frac{2}{n}}}{6n!} \pi^{\frac{n}{2}-\frac{3}{2}} \frac{n^3-2}{(n-1)(n+2)} n^{n+\frac{2}{n}-2} \frac{\Gamma(n+\frac{2}{n})\Gamma(\frac{n}{2})^{n+\frac{2}{n}}\Gamma(\frac{n^2+1}{2})}{\Gamma(\frac{n^2}{2})\Gamma(\frac{n+1}{2})^n}$$

# Take Home Message

- **Dimension 2:**
  - $\hookrightarrow$ Link between mean number of vertices and Gaussian curvature
  - $\hookrightarrow$ Result available for surface of negative curvature (Isokawa 2000)
  - $\hookrightarrow$ Other mean characteristics: area, perimeter
- **Dimension *n*:**
  - $\hookrightarrow$ Link between mean number of vertices and scalar curvature
  - $\hookrightarrow$ Perspective: other characteristics to get other curvatures

# Thank you for your attention!