Efficient nonparametric inference for discretely observed compound Poisson processes

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CCA

EPSRC
Engineering and Physical Sciences Research Council

Outline

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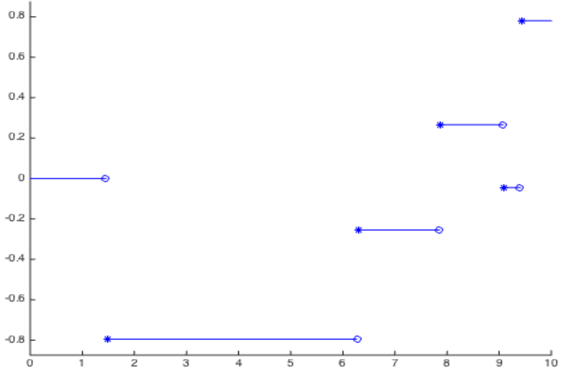

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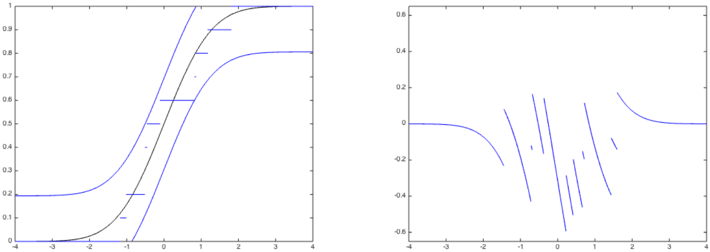
How to make inference on λ and F? Conditioned on $N_T = n$,

- λ can be estimated (parametrically) using the *maximum likelihood estimator* (asymptotic normality), and
- F can be estimated (nonparametrically) by the *empirical distribution function* $F_n(x) := \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{(-\infty,x]}(X_k)$, $x \in \mathbb{R}$.

Properties of $F_n(x) := \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{(-\infty,x]}(X_k)$



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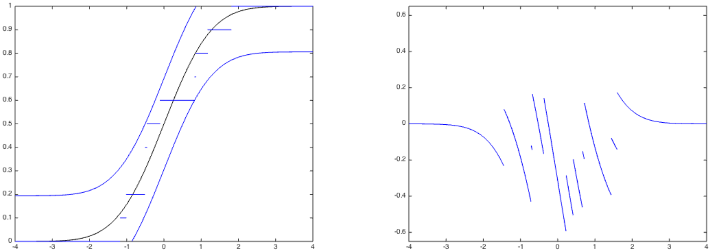

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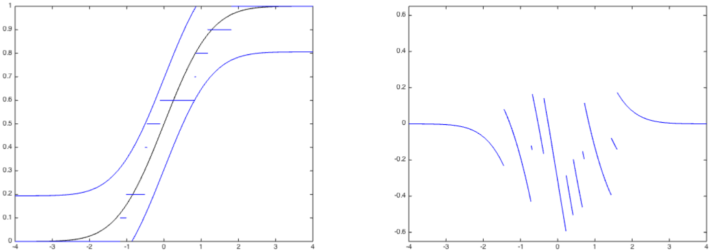

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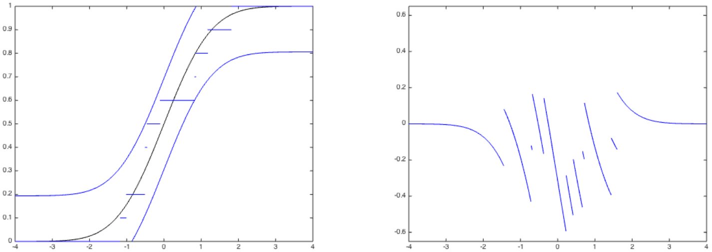

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Donsker's theorem: $\sqrt{n}\,(F_n - F) \to^{\mathcal{D}} \mathbb{G}_F$ in $L^\infty(\mathbb{R})$ as $n \to \infty$, where $\mathbb{G}_F$ is the mean-zero Gaussian process on $\mathbb{R}$ with covariance function $E[\mathbb{G}_F(x)\mathbb{G}_F(y)] = F(x \wedge y) - F(x)F(y)$.

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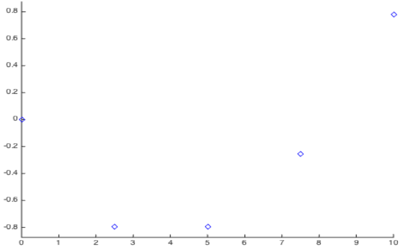

Figure: $C_\Delta, \ldots, C_{n\Delta}$, $\Delta = 2.5$ and $n = 4$ $(\lambda = 0.5, F = N(0,1))$

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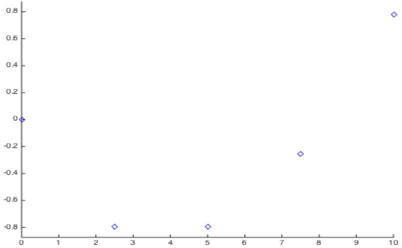

Figure: $C_\Delta, \ldots, C_{n\Delta}$, $\Delta = 2.5$ and $n = 4$ ($\lambda = 0.5, F = N(0,1)$)

How can we infer λ and F with such incomplete information?

CPPs as LPs and nonlinear inverse problem

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Answer: Yes! We resort to the spectral approach to find a heuristic reason and to construct such an estimator.

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The spectral approach: No ill-posedness and estimators

Let $\mathcal{N} := \lambda F$ be the *Lévy distribution* of C, denote the *Fourier transform* operator by $\mathcal{F}$ and let P be the law Y.
Then, the characteristic function of each increment ($\sim$ the noise) is

$$\varphi(u) := \mathcal{F}[P](u) := E[e^{iuY}] = e^{\Delta(\mathcal{F}[d\mathcal{N}](u) - \lambda)}, \quad u \in \mathbb{R}.$$

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Due to $\lim_{x\to\infty}\mathcal{N}(x) = \lambda$, $\lambda_n := \mathcal{N}_n(\infty)$ and $\widehat{F}_n := \mathcal{N}_n/\lambda_n$.

The spectral approach: $\widehat{F}_n := \mathcal{N}_n/\lambda_n$

How does $\widehat{F}_n$ and its $\sqrt{n}$-fluctuations about F look?

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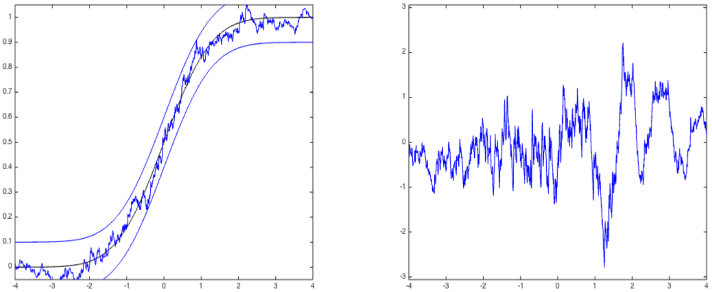

Left figure: Black: $F = N(0,1)$; Blue: $\widehat{F}_n, n = 10, 20, \ldots, 500$ and $F \pm \max|\widehat{F}_n - F|$

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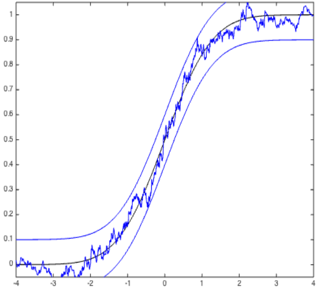

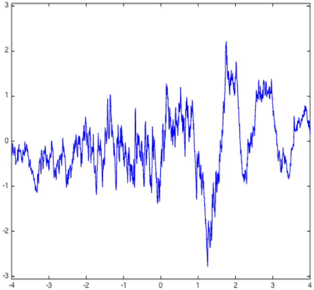

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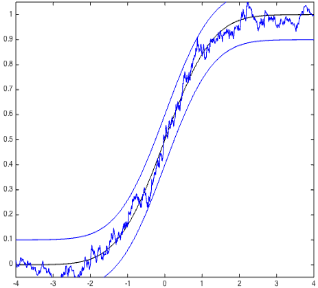

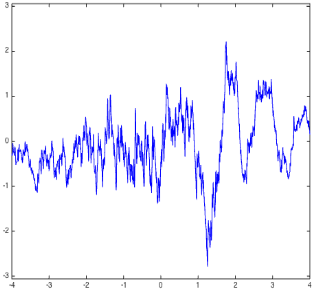

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Can we show an analogue of Donsker's theorem for $\widehat{F}_n$?

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Theorem (Coca (2015))

Take $h_n \sim \exp(-n^{\vartheta_h})$, $\varepsilon_n \sim \exp(-n^{\vartheta_\varepsilon})$ *and* $H_n \sim \exp(n^{\vartheta_H})$ *with* $0 < \vartheta_\varepsilon, \vartheta_H < \vartheta_h < 1/4$.

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$$\sqrt{n}\left(\widehat{F}_n - F\right) \to^{\mathcal{D}} \widehat{\mathbb{G}}_{\mathcal{N}} \quad \text{in } L^\infty(\mathbb{R}),$$

where $\widehat{\mathbb{G}}_{\mathcal{N}}$ *is a zero-mean Gaussian process on* $\mathbb{R}$ *with covariance function* $\Sigma_{x,y} := G(f_x, f_y)$*, with* $f_x := \lambda^{-1}\left(\mathbb{1}_{(-\infty,x]} - F(x)\right)\mathbb{1}_{\mathbb{R}\setminus\{0\}}$.

A Donsker theorem for discrete observations: Covariance

The covariances are defined through

$$G(g_1,g_2) := \frac{1}{\Delta^2}\int_{\mathbb{R}} (g_1 * \mathcal{F}^{-1}[1/\varphi(-\cdot)](x))(g_2 * \mathcal{F}^{-1}[1/\varphi(-\cdot)](x))P(dx),$$

where $*$ denotes the convolution operation, $\mathcal{F}^{-1}\left[\varphi^{-1}(-\cdot)\right]$ is a finite signed measure on $\mathbb{R}$ satisfying

$$\mathcal{F}^{-1}\left[\varphi^{-1}(-\cdot)\right] = e^{\Delta\lambda}\sum_{k=0}^{\infty}\bar{\nu}^{*k}\frac{(-\Delta)^k}{k!},$$

with $\bar{\nu}(A) := \nu(-A)$ for all $A \subseteq \mathbb{R}$ Borel measurable, $\bar{\nu}^{*k}$ is the k-fold convolution of $\bar{\nu}$ and $\bar{\nu}^{*0} = \delta_0$, and the probabillity measure P has the well-known representation (see Remark 27.3 in Sato (1999))

$$P = e^{-\Delta\lambda}\sum_{k=0}^{\infty}\nu^{*k}\frac{\Delta^k}{k!}.$$

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- (Optimal) confidence bands and goodness-of-fit tests for F can be constructed boostrap techniques such as substituting f, F, φ and P by their empirical counterparts in G;
- $\Delta\lambda\,\Sigma_{x,y} = F(x \wedge y) - F(x)F(y) + O(\Delta\lambda)$ so when $\Delta\lambda$ is small classical Donsker's theorem is recovered and these procedures can be approximated by analogues independent of F, φ and P.

A Donsker theorem for discrete observations: Proof

- Write $\sqrt{n}\left(\widehat{F}_n - F\right) = \lambda_n^{-1}\sqrt{n}\left((\mathcal{N}_n - \mathcal{N}) + F(\lambda - \lambda_n)\right)$. Therefore we need joint convergence of λ_n and $\mathcal{N}_n$ and, in particular, convergence of the latter:

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- The linear term is $(P_n - P)\,\psi_{x,n}$, where $Q\,\psi = \int_{\mathbb{R}}\psi\, dQ$ and $\psi_{x,n} := f_{x,n} * \mathcal{F}^{-1}[1/\varphi(-\cdot)] * K_{h_n}$. This is an empirical process indexed by a class of functions changing with n so, to show it is P-Donsker, we use the following theorem.

A Donsker theorem for discrete observations: Proof

Theorem (Theorem 2.11.23 in van der Vaart and Wellner (1996))

For each n, let $\Psi_n := \{\psi_{x,n} : x \in \mathbb{R}\}$ *be a class of measurable functions indexed by a totally bounded semimetric space* $(\mathbb{R}, \rho)$.

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$$\int_0^{\delta_n} \sqrt{\log N_{[]}(\epsilon \|\Psi_n\|_{L^2(P)}, \mathit{\Psi}_n, L^2(P))}\, d\epsilon, \sup_{\rho(x,y)<\delta_n} P(\psi_{x,n} - \psi_{y,n})^2 \to 0$$

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where $\|\psi\|_{L^2(P)} = (\int_{\mathbb{R}} |\psi|^2 P)^{1/2}$. *Then* $\left(\sqrt{n}\,(P_n - P)\,\psi_{x,n}\right)_{x\in\mathbb{R}}$ *is asymptotically tight in* $L^\infty(\mathbb{R})$ *and converges in distribution to a tight Gaussian process provided the sequence of covariance functions* $P\psi_{x,n}\psi_{y,n} - P\psi_{x,n}P\psi_{y,n}$ *converges pointwise on* $\mathbb{R} \times \mathbb{R}$.

A Donsker theorem for discrete observations: Extensions

In Coca (2015) we also show

- a Donsker theorem for $\mathcal{N}$. The limit process, of Brownian motion-type, is different from that of F, of Brownian bridge-type. The former provides optimal inference procedures for the CPP as a whole and gives insight into efficiency issues;

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Furthermore, the results can be extended to X being multidimensional and to noisy (unknown but observed noise) and nonequispaced discrete observations. Future manuscript?

Outline

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Thanks for your attention!