From the Magnetization Integral Equation to Cosserat and Stokes Eigenvalues

Histories of selfadjoint extensions

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A peculiar spectral structure

$\Omega \subset \mathbb{R}^d$ bounded domain. $A : L^2(\Omega) \to L^2(\Omega)$ bounded linear.

Essential spectrum: $\mathrm{Sp}_{\mathrm{ess}}(A) \subset [0,1]$ with:

1. 0 and 1 eigenvalues of infinite multiplicity
2. $\frac{1}{2}$ limit point of eigenvalues
3. if $\partial\Omega$ smooth: $\mathrm{Sp}_{\mathrm{ess}}(A) = \{0,1\} \cup \{\frac{1}{2}\}$
4. if $\partial\Omega$ Lipschitz: $\exists\delta \in (0, \frac{1}{2}) : \mathrm{Sp}_{\mathrm{ess}}(A) \subset \{0,1\} \cup [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$
5. if $\Omega \subset \mathbb{R}^2$ polygon: $\exists\delta \in (0, \frac{1}{2}) : \mathrm{Sp}_{\mathrm{ess}}(A) = \{0,1\} \cup [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$

Example 1 : The magnetization integral equation

[Friedman-Pasciak 1984] (Electrodynamics: [Co-Darrigrand-Koné-Sakly 2007–15])
Magnetostatics in $\mathbb{R}^3$

$$\operatorname{div} \boldsymbol{B} = 0 \,; \quad \mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \,; \quad \boldsymbol{B} = \mu \boldsymbol{H} \,;\, \mu = 1 \text{ outside of } \Omega$$

If $\mu \equiv 1$, $\operatorname{div} \boldsymbol{J} = 0$, $\operatorname{supp} \boldsymbol{J}$ compact: $\boldsymbol{H}^{\rm in} := \mathbf{curl}\, g_0 * \boldsymbol{J} \,;\quad g_0(x) = \frac{1}{4\pi|x|}$
Magnetization: $\boldsymbol{B} - \boldsymbol{H} = \nu \boldsymbol{B}$, $\nu = 1 - \frac{1}{\mu}$

$$\operatorname{div} \boldsymbol{B} = 0 \,; \quad \mathbf{curl}\, \boldsymbol{B} = \boldsymbol{J} + \mathbf{curl}(\boldsymbol{B} - \boldsymbol{H}) \quad \Longrightarrow \quad \boldsymbol{B} = \boldsymbol{H}^{\rm in} + \mathbf{curl}\, g_0 * \mathbf{curl}\, \nu \boldsymbol{B}$$

$$\mathbf{curl}\,\mathbf{curl}\, g_0 * - \nabla \operatorname{div} g_0 * = 1$$

Volume integral equation ($\mu \in \mathbb{C}$ in Ω) $\boxed{\boldsymbol{H} - (1-\mu) A\boldsymbol{H} = \boldsymbol{H}^{\rm in}}$
Volume integral operator

$$\boxed{A\boldsymbol{u}(x) = -\nabla \operatorname{div} \int_\Omega g_0(x-y)\boldsymbol{u}(x)dy}$$

Strongly singular! VIE Fredholm? $\to \operatorname{Sp}_{\rm ess}(A)$

Example 1.1 : VIE in acoustic scattering

$$\operatorname{div} a(x)\nabla u + k(x)^2 u = f \qquad \text{in } \mathbb{R}^d$$

$a(x) \equiv 1, \quad k(x) \equiv k \in \mathbb{C}$ outside of the bounded domain Ω.

Rewritten as perturbation problem

$$(\Delta + k^2)u = f - \operatorname{div} \alpha \nabla u - \beta u$$

with

$$\alpha(x) = a(x) - 1, \; \beta(x) = k(x)^2 - k^2 .$$

Convolution with the Helmholtz fundamental solution $g_k(x) = \frac{e^{ik|x|}}{4\pi|x|}$ $(d = 3)$

Volume integral equation

$$\boxed{u(x) - \operatorname{div} \int_\Omega g_k(x-y)\alpha(y)\nabla u(y)\,dy - \int_\Omega g_k(x-y)\beta(y)u(y)\,dy = u^{\mathrm{in}}(x)}$$

Modulo lower order (compact) operators, volume integral operator

$$\boxed{A_0 u(x) = -\operatorname{div} \int_\Omega g_0(x-y)\nabla u(y)\,dy}$$

Example 2 : The Cosserat eigenvalue problem

[E.&F. Cosserat 1898]

$$\sigma\Delta \boldsymbol{u} - \nabla \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\,; \quad \boldsymbol{u} = 0 \text{ on } \partial\Omega$$

Motivation:
Eigenfunction expansion for solving the Lamé system of elasticity.

- Spectral decomposition of the operator

$$\Delta^{-1}\nabla \operatorname{div} \text{ in } \boldsymbol{H}_0^1(\Omega)$$

- Or equivalently ($Sp(ST) = Sp(TS)$ in $\mathbb{C}\setminus\{0\}$), spectral decomposition of the operator

$$\boxed{S = \operatorname{div}\Delta^{-1}\nabla} \quad \text{in } L^2_\circ(\Omega) = \{u \in L^2(\Omega) \mid \int_\Omega u = 0\}$$

The eigenvalue problem $Sp = \sigma p$ becomes, with $\boldsymbol{u} = \Delta^{-1}\nabla p$, the system

$$\Delta\boldsymbol{u} - \nabla p = 0\,;\ \operatorname{div}\boldsymbol{u} = \sigma p.$$

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Example 2.1 : Two Stokes eigenvalue problems

Stokes eigenvalue problem, first kind

Find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$, $p \in L^2_\circ(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$:

$$\begin{array}{lll} -\Delta \boldsymbol{u} + \nabla p = \sigma \boldsymbol{u} & & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} & = 0 & \text{in } \Omega \end{array}$$

Stokes eigenvalue problem, second kind

Find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$, $p \in L^2_\circ(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$:

$$\begin{array}{lll} -\Delta \boldsymbol{u} + \nabla p = 0 & & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} & = \sigma p & \text{in } \Omega \end{array}$$

1st kind:

- Elliptic eigenvalue problem, compact resolvent,
- Known conditions for convergence of numerical algorithms (discrete LBB condition...)
- Appears in dynamic problems (time stepping, Laplace transform)

2nd kind:

- No compact resolvent, infinite-dimensional eigenspace for $\sigma = 1$ ($p \in \Delta C_0^\infty(\Omega)$, $\boldsymbol{u} = \nabla \Delta^{-1} p = \Delta^{-1} \nabla p$)
- No convergent numerical algorithm known
- Equivalent to Cosserat eigenvalue problem. S: Schur complement.

Example 2.2 : The first Stokes eigenvalue problem and LBB

Definition: Ladyzhenskaya-Babuška-Brezzi constant or inf-sup constant

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q}{|\boldsymbol{v}|_1 \, \|q\|_0}$$

$$\|q\|_0 = \|q\|_{L^2(\Omega)} \; ; \quad |v|_1 = \|\nabla v\|_0$$

Conforming approximation: $X_N \subset H^1_0(\Omega)$, $M_N \subset L^2_\circ(\Omega)$, $N \to \infty$
Define discrete LBB constant $\beta_N(\Omega)$ analogously.

Theorem

Under usual approximation assumptions, the discrete LBB condition

$$\forall N: \; \beta_N(\Omega) \geq \beta_*(\Omega) > 0$$

is necessary and sufficient for spectrally correct approximation of the first kind Stokes eigenvalue problem.

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Theorem [Brezzi-Boffi-Gastaldi 1997]

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is necessary and sufficient for spectrally correct approximation of the first kind Stokes eigenvalue problem.

The second kind Stokes (Cosserat) eigenvalues determine the LBB constant

Let $\sigma(\Omega) = \min \mathrm{Sp}(S)$. Then

$$\sigma(\Omega) = \beta(\Omega)^2$$

Proof:

$-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Riesz isometry.

Let $q \in L^2_\circ(\Omega)$.

$$\begin{aligned}\langle Sq, q\rangle &= \langle \operatorname{div} \Delta^{-1} \nabla q, q\rangle \\ &= \langle -\Delta^{-1} \nabla q, \nabla q\rangle \\ &= \|\nabla q\|^2_{-1} \\ &= \Big(\sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{\langle \nabla q, \boldsymbol{v}\rangle}{|\boldsymbol{v}|_1}\Big)^2 = \Big(\sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{\langle q, \operatorname{div} \boldsymbol{v}\rangle}{|\boldsymbol{v}|_1}\Big)^2\end{aligned}$$

$$\sigma(\Omega) = \inf_{q\in L^2_\circ(\Omega)} \frac{\langle Sq, q\rangle}{\langle q, q\rangle} = \beta(\Omega)^2$$

Spectrum of the magnetization integral operator

Orthogonal decomposition into invariant subspaces:

$$\boldsymbol{L}^2(\Omega) = \nabla H^1_0(\Omega) \oplus V\,; \qquad V = H(\operatorname{div} 0, \Omega)$$

$$= \nabla H^1(\Omega) \oplus V_0\,; \qquad V_0 = H_0(\operatorname{div} 0, \Omega)$$

$$= \nabla H^1_0(\Omega) \oplus V_0 \oplus W\,; \qquad W = \nabla H^1(\Omega) \cap V \text{ : harmonic vector fields}$$

Recall: $\quad A\boldsymbol{u}(x) = -\nabla \operatorname{div} g_0 * (\chi_\Omega \boldsymbol{u})(x) = \nabla_x \int_\Omega \nabla_y \frac{1}{4\pi|x-y|} \cdot u(y)\,dy$

Lemma [illegible]

$$\boldsymbol{u} \in \nabla H^1_0(\Omega) \Longrightarrow \quad A\boldsymbol{u} = \boldsymbol{u}$$

$$\boldsymbol{u} \in V_0 \Longrightarrow \quad A\boldsymbol{u} = 0$$

$$\boldsymbol{u} \in W \Longrightarrow \quad A\boldsymbol{u} = \nabla \mathscr{S}(\gamma_n \boldsymbol{u}) \in W$$

$\mathscr{S}$: harmonic single layer potential

Isomorphisms (normal trace and solution of Neumann problem):

$$W \ni u \leftrightarrow \gamma_n u = n \cdot u\big|_{\partial\Omega} \in H_*^{-1/2}(\partial\Omega)$$

$$A\big|_W \leftrightarrow \partial_n S = \left(\tfrac{1}{2} + K'\right)\big|_{H_*^{-1/2}(\partial\Omega)}$$

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Spectrum of the magnetization integral operator

Corollary (VIO $\longrightarrow$ double layer BIO)

$$\mathrm{Sp}_{\mathrm{ess}}(A) = \{0,1\} \cup \mathrm{Sp}_{\mathrm{ess}}(\tfrac{1}{2} + K')$$

Similarly, for the acoustic scattering VIO $A_0 = -\operatorname{div} \int_\Omega g_0(x-y) \nabla u(y)\, dy$:

$$\mathrm{Sp}_{\mathrm{ess}}(A_0) = \{0,1\} \cup \mathrm{Sp}_{\mathrm{ess}}(\tfrac{1}{2} + K)$$

Harmonic double layer potential boundary integral operator

$$Ku(x) = \int_{\partial\Omega} \partial_{n(y)} g_0(x-y)\, u(y)\, ds(y)$$

Classical facts about the spectrum of $\frac{1}{2} + K$ (in L^2 or $H^{\frac{1}{2}}$):

1. [Fredholm 1900] : If $\partial\Omega$ smooth: $\mathrm{Sp}_{\mathrm{ess}}(\frac{1}{2} + K) = \{\frac{1}{2}\}$
2. [Poincaré 1896], . . . , [Co 2007] : If $\partial\Omega$ Lipschitz, then in $H^{\frac{1}{2}}(\Omega) \cap L^2_\circ(\Omega)$
   $\mathrm{Sp}(\frac{1}{2} + K) \subset (0,1)$
3. [Co-Stephan 1981] If $\Omega \subset \mathbb{R}^2$ polygon: $\exists \delta \in (0, \frac{1}{2})$:
   $\mathrm{Sp}_{\mathrm{ess}}(\frac{1}{2} + K) = [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$

The operator B : Two different selfadjoint extensions

In the electrodynamic scattering problem, when both the electric permittivity and the magnetic permeability are piecewise constant, one finds the VIE

$$\boxed{\boldsymbol{E} - \eta A\boldsymbol{E} - \nu B\boldsymbol{E} = \mathbf{E}^{\mathrm{in}}}$$

with $\eta, \nu \in \mathbb{C}$ and the operators (modulo compact perturbations)

$$A\boldsymbol{u}(x) = -\nabla \operatorname{div} \int_\Omega g_0(x-y)\boldsymbol{u}(y)dy$$

$$B\boldsymbol{u}(x) = \mathbf{curl} \int_\Omega g_0(x-y)\,\mathbf{curl}\,\boldsymbol{u}(y)dy$$

On $C_0^\infty(\Omega)$, we have seen $A + B = 1$.

Define B_0 as the extension from $C_0^\infty(\Omega)$ to $L^2(\Omega)$ of the bounded operator B (ΨDO of order 0) $\Longrightarrow A + B_0 = 1$ on $L^2(\Omega)$.

Corollary

If $\partial\Omega$ is smooth, then $\quad \mathrm{Sp}_{\mathrm{ess}}(\eta A + \nu B_0) = \{\eta, \nu, \frac{\eta+\nu}{2}\}$

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The operator B : Two different selfadjoint extensions

However...

[Co-Darrigrand-Sakly 2011]
The "physical" operator B has $H(\mathbf{curl}, \Omega)$ as its domain of definition and is unbounded in L^2

$$B \neq B_0$$

The VIE $(1 - \eta A - \nu B_0)E = E^{in}$ in L^2 corresponds to a non-physical scattering problem.
The operators A and B have distinct invariant subspaces.

Theorem

If $\partial\Omega$ is smooth, then $\qquad \mathrm{Sp}_{ess}(\eta A + \nu B) = \{0, \frac{\eta}{2}, \eta, \nu, \frac{\nu}{2}\}$

...and now over to Cosserat...

$$(\lambda - A)\boldsymbol{u}(x) = -(\lambda\,\Delta - \nabla\,\mathrm{div})\int_\Omega \frac{\boldsymbol{u}(y)\,dy}{4\pi|x-y|}$$

Cosserat with a different boundary condition (radiation condition) !

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...and now over to Cosserat...

$$(\lambda - A)u(x) = -(\lambda\Delta - \nabla\operatorname{div})\int_\Omega \frac{u(y)\,dy}{4\pi|x-y|}$$

Cosserat with a different boundary condition (radiation condition) !

However...

[Co-Darrigrand-Sakly 2011]

The "physical" operator B has $H(\mathbf{curl},\Omega)$ as its domain of definition and is unbounded in L^2

$$B \neq B_0$$

The VIE $(1-\eta A-\nu B_0)\boldsymbol{E}=\boldsymbol{E}^{\rm in}$ in L^2 corresponds to a non-physical scattering problem.

The operators A and B have distinct invariant subspaces.

Theorem [Co-Darrigrand-Sakly 2011]

If $\partial\Omega$ is smooth, then $\quad \mathrm{Sp}_{\rm ess}(\eta A+\nu B)=\{0,\frac{\eta}{2},\eta,\nu,\frac{\nu}{2}\}$

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The Cosserat eigenvalue problem: Basics

Recall: The Cosserat eigenvalue problem

$$\sigma \Delta \boldsymbol{u} = \nabla \operatorname{div} \boldsymbol{u} \, ; \quad \boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$$

$\Longleftrightarrow$ Spectral problem for the operator

$$S = \operatorname{div} \Delta^{-1} \nabla \ \text{ in } L^2_\circ(\Omega) \, ; \quad \Delta = \Delta_{\mathrm{Dir}}$$

Classical result [Mikhlin 1973]

If $\partial\Omega$ is smooth ([Crouzeix 1997]: C^3), then

1. There is a o.n. basis of eigenfunctions
2. $\sigma = 1$ is an eigenvalue of infinite multiplicity
3. $\sigma = \frac{1}{2}$ is a limit point of eigenvalues

Example [E&F Cosserat 1898]

Ω ball in $\mathbb{R}^d$:

$$\sigma_k = \frac{k}{2k + d - 2} \, , \; k \geq 1 \; : \quad \frac{1}{d} \nearrow \frac{1}{2}$$

Cosserat eigenvalue problem: Basics

Simple observations

For any bounded domain:

1. $\mathrm{Sp}(S) \subset [0,1]$
2. 1 is ev of ∞ multiplicity: $p \in \Delta H_0^2(\Omega) \Rightarrow Sp = p$
3. $\mathrm{Sp}(S)$ is invariant under translations, rotations, dilations
4. In $d = 2$, $\mathrm{Sp}(S)$ is symmetric wrt $\frac{1}{2}$
   $\mathbf{curl}\,\mathrm{curl} = r \circ (\nabla\,\mathrm{div}) \circ r^{-1}$, r rotation by 90°, and
   $\sigma\Delta - \nabla\,\mathrm{div} = -\big((1-\sigma)\nabla\,\mathrm{div} + \sigma\,\mathbf{curl}\,\mathbf{curl}\big)$

Main quantity of interest: $\sigma(\Omega) = \min \mathrm{Sp}(S)$. Bigger is better

[Friedrichs 1937], … [Babuška 1971]: Ω Lipschitz $\Rightarrow \sigma(\Omega) > 0$

Best result

Ω bounded John domain $\Rightarrow \sigma(\Omega) > 0$

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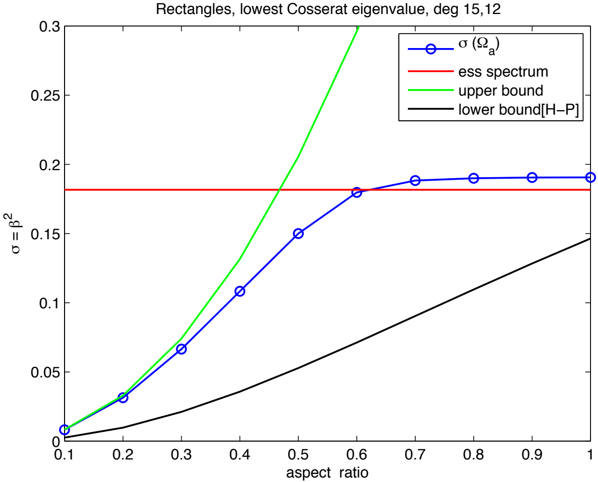
Best result [Acosta-Durán-Muschietti 2006], [Durán 2012]

Ω bounded John domain $\Rightarrow \sigma(\Omega) > 0$

▸ Digression: Starshaped and John domains

Computation on rectangles with aspect ratio $0.1 \dots 1$
80 elements (Q_{15}, Q_{12}), $\sim$ 30000 dof
First Cosserat eigenvalue (computed with a Stokes solver)

- $\sigma(\Omega) = \beta(\Omega)^2$ is the minimum of the Cosserat spectrum



Cosserat on the rectangle: Upper bounds

Trying to find an asymptotic expansion as $a = \varepsilon \to 0$:
Change of scale $y \mapsto \frac{y}{\varepsilon}$: $\Omega_\varepsilon \mapsto$ square.
Insert into Cosserat or Stokes eigenvalue problem.
Expand in powers of ε.

No luck so far..

But: The leading term in the expansion suggests a (quasi-)mode where

$$p(x,y) = \cos(\pi x)$$

and $\Delta^{-1}\nabla p$ can be computed explicitly:

$$\boldsymbol{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}; \quad u(x,y) = \frac{1}{\pi}\sin\pi x\,\Big(1 - \frac{\cosh \pi y}{\cosh\frac{\pi\varepsilon}{2}}\Big)$$

This leads to an upper bound $\sigma(\Omega_\varepsilon) \le \frac{|u|_1^2}{\|p\|_0^2} = 1 - \frac{2}{\pi\varepsilon}\tanh\frac{\pi\varepsilon}{2}$

$$\sigma(\Omega_\varepsilon) \le \frac{\pi^2}{12}\varepsilon^2$$

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Cosserat on the rectangle: Lower bounds

The Horgan-Payne inequality [Horgan-Payne 1983], [Co-Dauge 2015]

Ω star-shaped wrt a ball.
Define $\omega =$ minimal angle between radius vector and tangent at the boundary. Then ([Co-Dauge]: for some domains..., in particular rectangle)

$$\sigma(\Omega) \geq \sin^2 \tfrac{\omega}{2}$$

For the rectangle Ω_ε

$$\sigma(\Omega_\varepsilon) \geq \frac{\varepsilon^2}{4(1+\varepsilon^2+\sqrt{1+\varepsilon^2})} \geq \frac{\varepsilon^2}{7}$$

Cosserat on the rectangle: Singularities

New phenomenon:

The eigenfunctions have corner singularities $\boldsymbol{u} \sim r^{\lambda}$ at the corners, and their strength (exponent of singularity λ) depends on the eigenvalue σ. The situation $\operatorname{Re}\lambda(\sigma) = 0$ corresponds to the continuous spectrum.

The exponent λ can be found with Kondrat'ev's method. It satifies

$$(1-2\sigma)\frac{\sin\lambda\omega}{\lambda} = \pm\sin\omega.$$

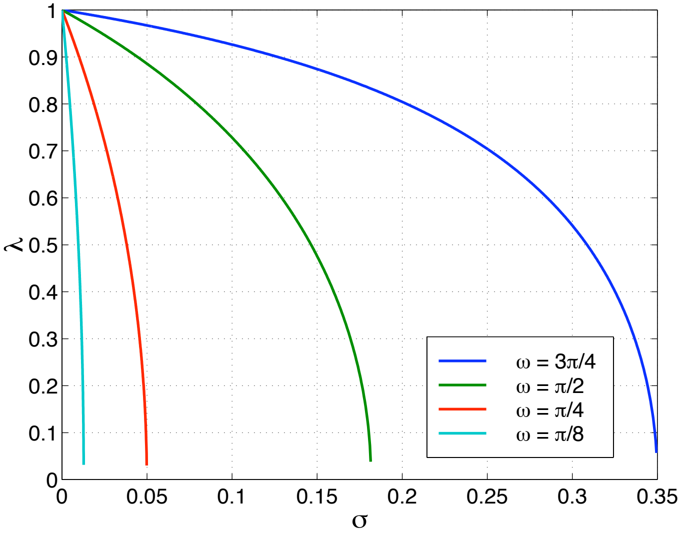
Result [Co-Crouzeix-Dauge-Lafranche 2015]

A corner angle ω contributes to $\mathrm{Sp}_{\mathrm{ess}}(S)$ the interval

$$[\tfrac12-\delta, \tfrac12+\delta] \text{ with } \delta = \frac{\sin\omega}{\omega}$$

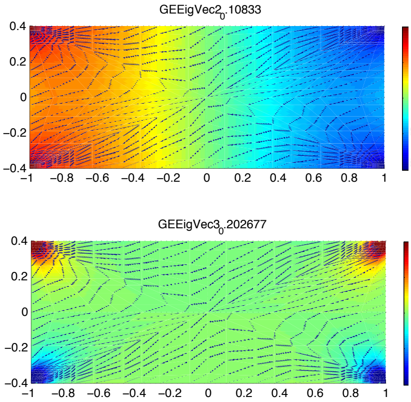
Corollary for the rectangle: $\mathrm{Sp}_{\mathrm{ess}}(S) = [\frac12 - \frac1\pi, \frac12 + \frac1\pi]$

Remark: This is not the same interval as for $\mathrm{Sp}_{\mathrm{ess}}(\frac12 + K)$.

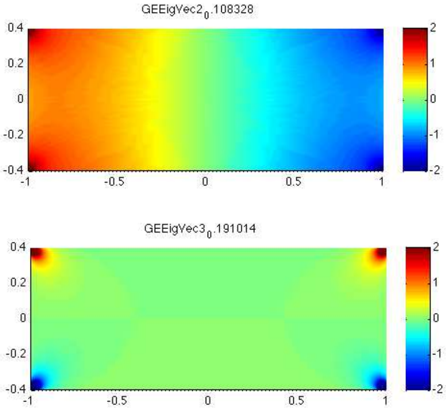

1
0.9
0.8
0.7
0.6
λ 0.5
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
σ
ω = 3π/4
ω = π/2
ω = π/4
ω = π/8

Rectangle, aspect ratio 0.4

Degrees: $6,3$

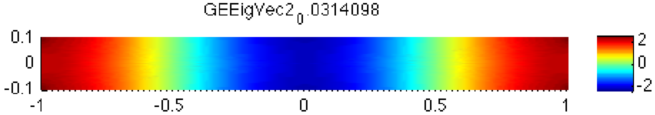


Degrees: 15,12

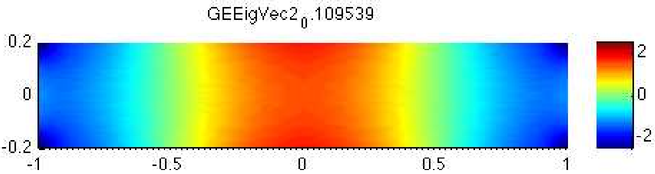


Rectangle: First eigenfunction

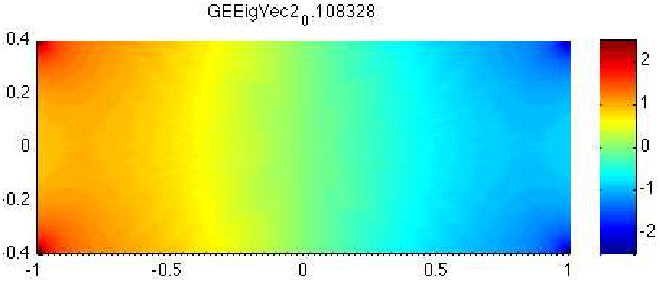
Rectangle, aspect ratio 0.1



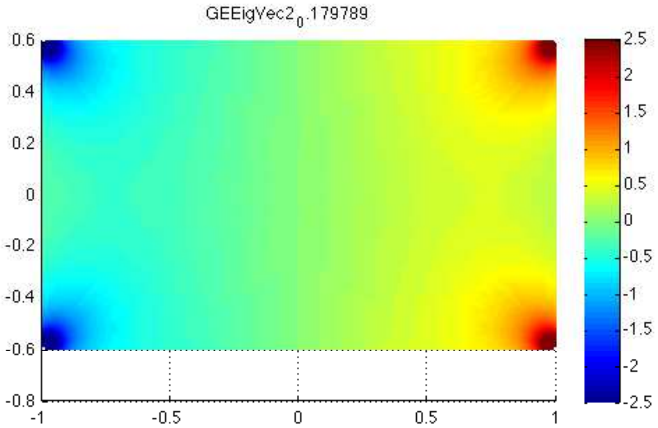
Rectangle, aspect ratio 0.2



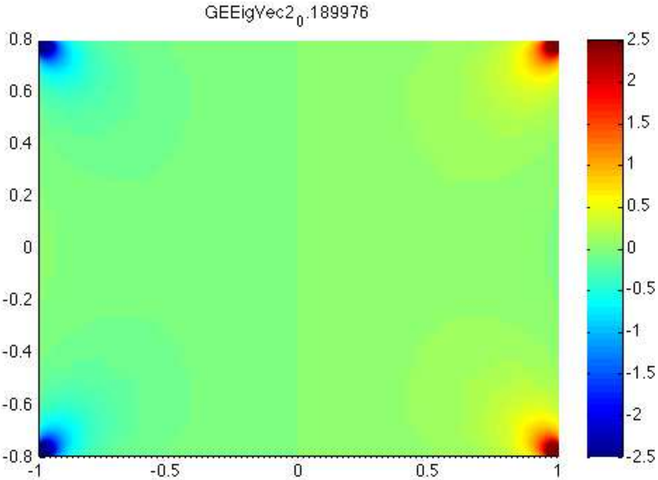
Rectangle, aspect ratio 0.4

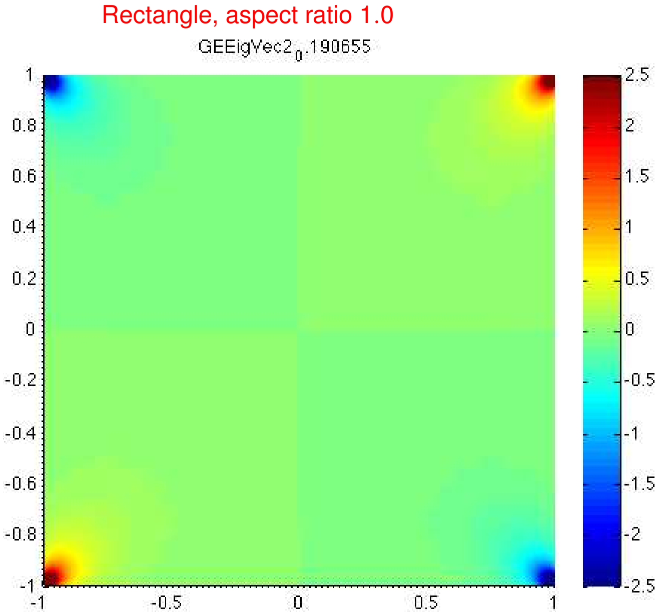


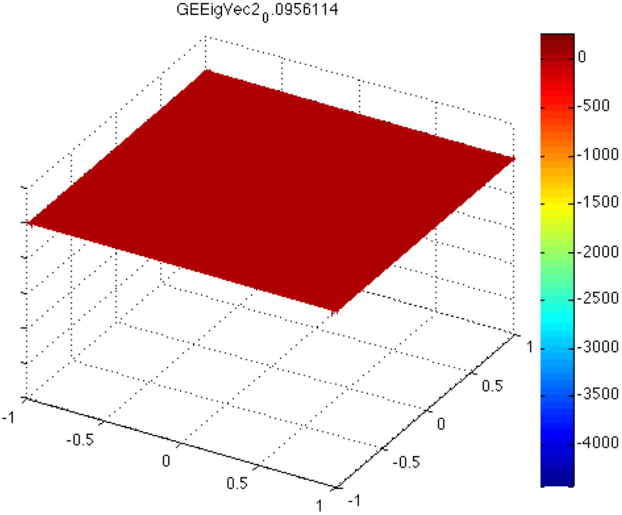
Rectangle, aspect ratio 0.6



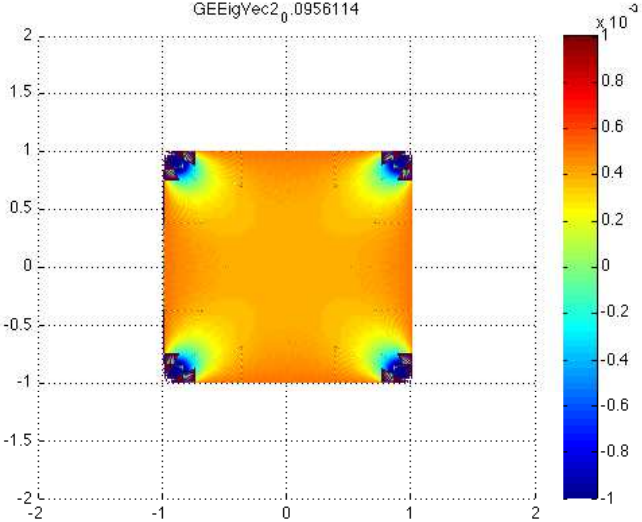
Rectangle, aspect ratio 0.8




Rectangle, aspect ratio 1.0
GEEigVec2$_0$.190655
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5


GEEigVec2_0.0956114
0
-500
-1000
-1500
-2000
-2500
-3000
-3500
-4000
-1
-0.5
0
0.5
1
-1
-0.5
0
0.5
1

Square: First eigenfunction, (Q_{17}, Q_{16})



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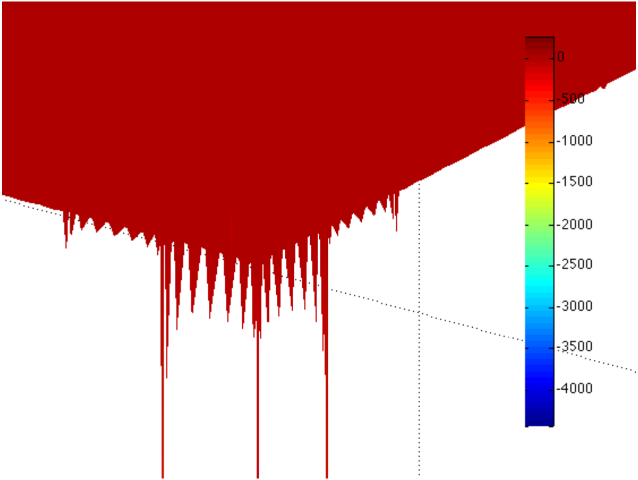

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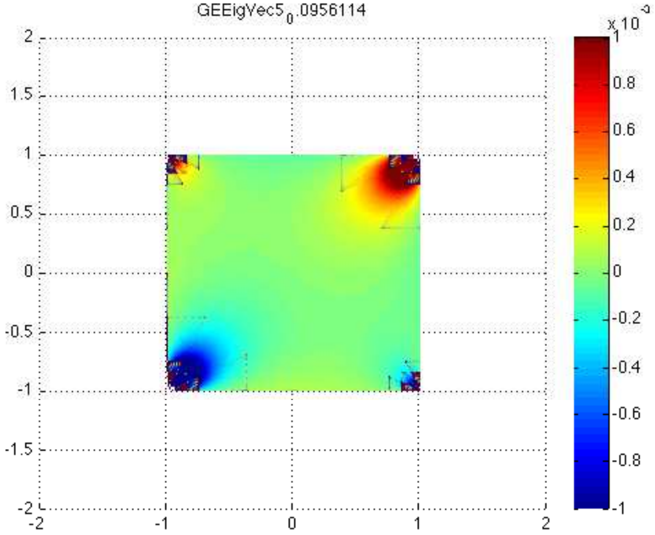

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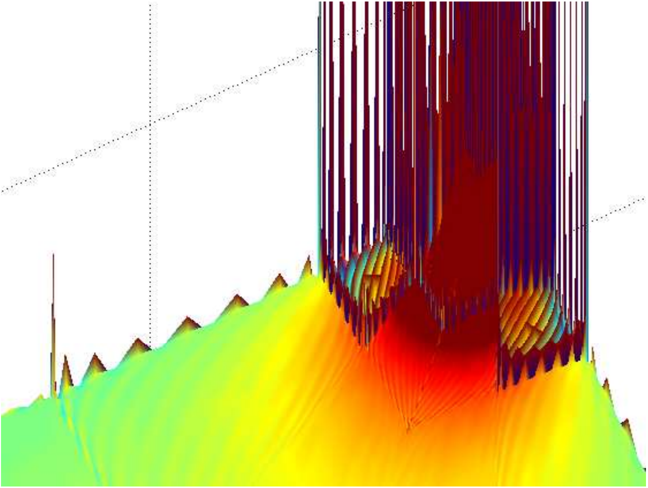

The Cosserat spectrum in 2D is always symmetric around $\frac{1}{2}$.
In the case of a polygon, $\mathrm{Sp}_{\mathrm{ess}}(\frac{1}{2}+K)$ is a symmetric interval around $\frac{1}{2}$.
Is there always symmetry of $\mathrm{Sp}_{\mathrm{ess}}(\frac{1}{2}+K)$ in 2D?

Theorem

For any bounded Lipschitz domain in $\mathbb{R}^2$:

$$\mathrm{Sp}_{\mathrm{ess}}(\tfrac{1}{2}+K) = \mathrm{Sp}_{\mathrm{ess}}(\tfrac{1}{2}-K)$$

The proof uses the equivalence with a scalar transmission problem.

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Theorem [Co-Darrigrand-Sakly 2015]

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The proof uses the equivalence with a scalar transmission problem.

Miscellaneous: Other boundary conditions

Recall

$$S = \operatorname{div}\Delta^{-1}\nabla\,; \quad A = \nabla\operatorname{div}\Delta^{-1}\,; \quad A_0 = \operatorname{div}\Delta^{-1}\nabla$$

with various realizations of Δ^{-1}. Does this explain the spectral structure?

If one replaces $\boldsymbol{H}_0^1(\Omega)$ in the Cosserat or Stokes problem by

$$\boldsymbol{H}_T(\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{n}\cdot\boldsymbol{u} = 0 \text{ on } \partial\Omega\}$$

the spectral structure changes completely:

If the Neumann problem on Ω has H^2-regularity (for example if Ω is convex or smooth), then

$$\mathrm{Sp}(S) = \{1\}$$

i.e. the operator $S = \operatorname{div}\Delta^{-1}\nabla$ is the identity on $L^2_\circ(\Omega)$.

Proof: For $p \in L^2_\circ(\Omega)$, one has then

$$\Delta^{-1}_{\boldsymbol{H}_T}\nabla p = \nabla\,\Delta^{-1}_{\mathrm{Neu}}p$$

The condition $\sigma_{\boldsymbol{H}_T}(\Omega) = 1$ is also sufficient for H^2-regularity.

- Is it true that in any dimension d

$$\sigma(\Omega) \leq \frac{1}{d} = \sigma(\text{Ball}) \ ?$$

Conjecture

$\sigma(\Omega) = \frac{1}{2} - \frac{1}{\pi} = \min \mathrm{Sp}_{\mathrm{ess}}(S)$

Need to show: There are no eigenvalues below the continuous spectrum.

Open problems 2: The Square

- If Ω is a square, what is $\sigma(\Omega)$?

Conjecture

$\sigma(\Omega) = \frac{1}{2} - \frac{1}{\pi} = \min \mathrm{Sp}_{\mathrm{ess}}(S)$

Need to show: There are no eigenvalues below the continuous spectrum.

Open problems 3: Numerical approximation

The only approximation result known is for approximation of the domain Ω by Ω_N.

[Bernardi-Co-Dauge-Girault 2015]

If $F_N : \Omega \to \Omega_N$ is bi-Lipschitz and $F_N \to \mathrm{Id}$ in the Lipschitz norm, then $\sigma(\Omega_N) \to \sigma(\Omega)$.

In general, if $M_N \subset L^2_\circ(\Omega)$ and $X_N \subset \boldsymbol{H}^1_0(\Omega)$ in the second kind Stokes eigenvalue problem, and one supposes approximation properties of these spaces, one only gets upper semicontinuity

$$\limsup \sigma_N \leq \sigma(\Omega)$$

Open problem

Find a numerical algorithm for approximating the Cosserat spectrum at least for polygons in $\mathbb{R}^2$

Thank you for your attention!

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Starshaped domains

Theorem [Bogovskiĭ 1979], [Galdi 1994]

Let $\Omega \subset \mathbb{R}^n$ be contained in a ball of radius R, starshaped with respect to a concentric ball of radius ρ. There exists a constant γ_d only depending on the dimension d such that

$$\beta(\Omega) \geq \gamma_d \left(\frac{\rho}{R}\right)^{d+1}$$

In dimension $d = 2$, we can prove:

$$\beta(\Omega) \geq \frac{\rho}{2R}$$

M. COSTABEL, M. DAUGE: On the Inequalities of Babuška-Aziz, Friedrichs and Horgan-Payne. Arch. Rational Mech. and Anal. (2015).

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John domains

Theorem [Acosta-Durán-Muschietti 2006], [Durán 2012]

Let $\Omega \subset \mathbb{R}^d$ be a bounded John domain. Then $\beta(\Omega) > 0$.

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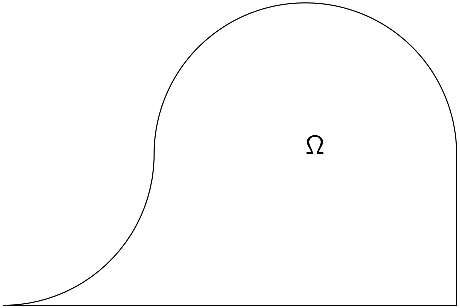

Figure: Not a John domain: Outward cusp, $\beta(\Omega) = 0$ [Friedrichs 1937]

Definition of John Domain

Definition

A domain $\Omega \subset \mathbb{R}^d$ with a distinguished point $\boldsymbol{x}_0$ is called a John domain if it satisfies the following "twisted cone" condition:
There exists a constant $\delta > 0$ such that, for any $\boldsymbol{y}$ in Ω, there is a rectifiable curve γ: $[0,\ell] \to \Omega$ parametrized by arclength such that

$$\gamma(0) = \boldsymbol{y}, \quad \gamma(\ell) = \boldsymbol{x}_0, \quad \text{and} \quad \forall t \in [0,\ell]: \ \ \mathrm{dist}(\gamma(t), \partial\Omega) \geq \delta\, t\,.$$

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Example : Every weakly Lipschitz domain is a John domain.

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Example : Every weakly Lipschitz domain is a John domain.

A John domain: Union of Lipschitz domains

San Juan de la Peña, Jaca 2013

A John domain: Zigzag

Figure: A weakly Lipschitz domain: the self-similar zigzag

John domains: Spirals

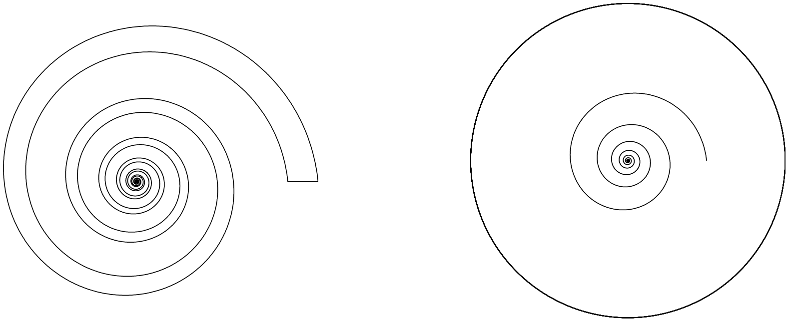


Figure: Weakly Lipschitz (left), John domain (right)

Fractal John domains: Tree or Lung

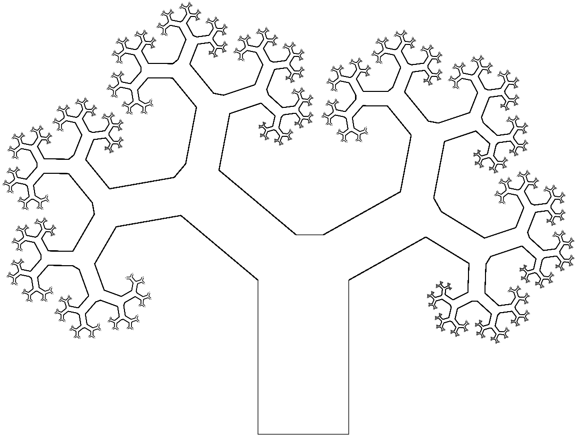
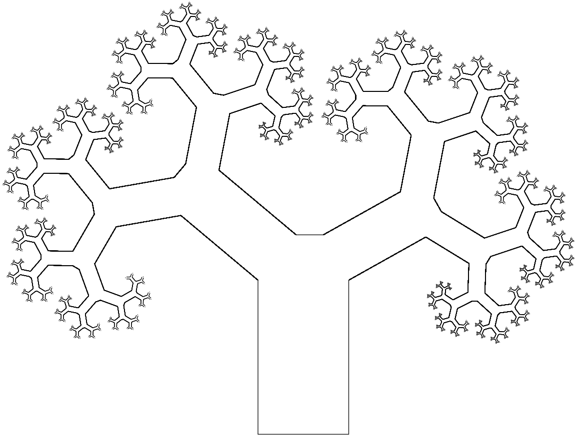


Figure: A John domain: the infinite tree

back

Fractal John domains: Tree or Lung



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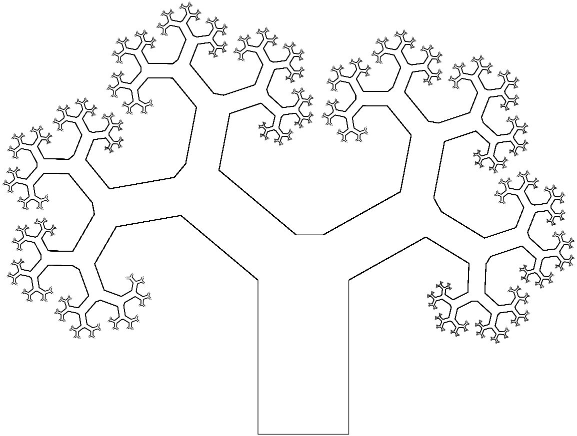
Fractal John domains: Tree or Lung



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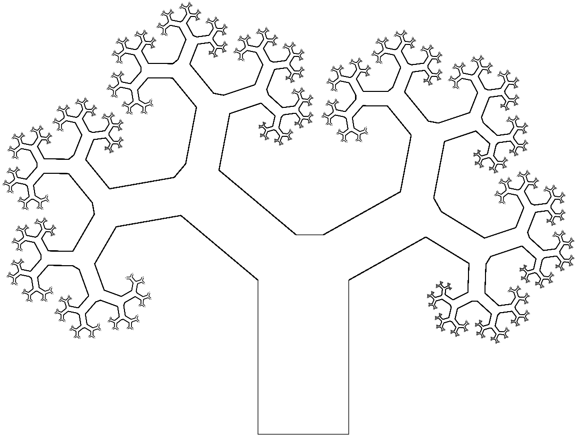

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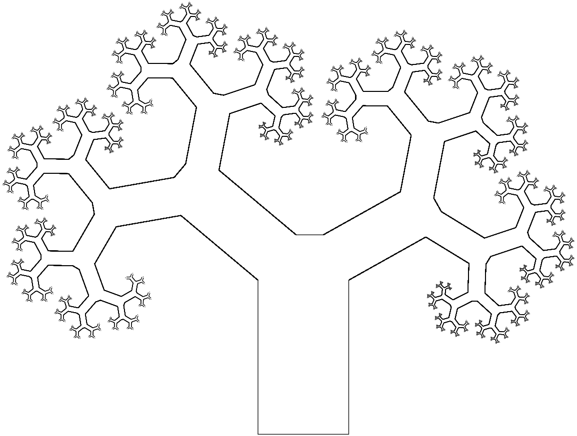

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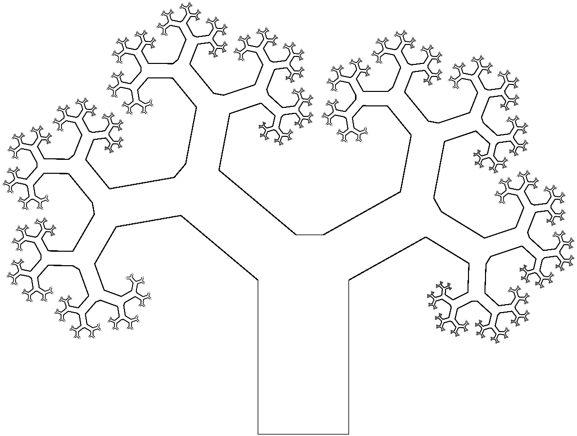

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