Semiclassical Magnetic Sobolev Constants

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Based on joint work with
Nicolas Raymond (Rennes).

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Set-up

Let $\Omega \subset \mathbb{R}^2$ be bounded and $p > 2$. Consider the quadratic form on $\mathrm{H}^1_0(\Omega)$ (Dirichlet bdry conditions),

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) = \int_\Omega |(-ih\nabla + \mathbf{A})\psi|^2 \, \mathrm{d}\mathbf{x},$$

and define the 'non-linear eigenvalue',

$$\lambda(\Omega, \mathbf{A}, p, h) = \inf_{\psi \in \mathrm{H}^1_0(\Omega), \psi \neq 0} \frac{\mathcal{Q}_{h,\mathbf{A}}(\psi)}{\left(\int_\Omega |\psi|^p \, \mathrm{d}x\right)^{\frac{2}{p}}} = \inf_{\substack{\psi \in \mathrm{H}^1_0(\Omega), \\ \|\psi\|_{\mathrm{L}^p(\Omega)} = 1}} \mathcal{Q}_{h,\mathbf{A}}(\psi).$$

How does $\lambda = \lambda(\Omega, \mathbf{A}, p, h)$ behave as $h \to 0_+$?

Of course, when $p = 2$ we get the ground state energy for the Dirichlet realization $H_{h,\mathbf{A}}$ of $(-ih\nabla + \mathbf{A})^2$.

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Basics

Lemma

The infimum is a minimum. Furthermore, any minimizer ψ *satisfies,*

$$(-ih\nabla + \mathbf{A})^2\psi = \|\psi\|_p^{2-p}\lambda(\Omega, \mathbf{A}, p, h)|\psi|^{p-2}\psi$$

(Euler-Lagrange equation).

Follows easily since $H_0^1(\Omega) \to L^p(\Omega)$ is compact.

We denote by $\mathbf{B}$ the magnetic field, i.e. $\mathbf{B} = \nabla \times \mathbf{A} = \partial_1 A_2 - \partial_2 A_1$.

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Models

For $k \in \mathbb{N}$, we define

$$\lambda^{[k]}(p) = \lambda(\mathbb{R}^2, \mathbf{A}^{[k]}, p, 1) = \inf_{\psi \in \mathrm{Dom}(Q_{\mathbf{A}^{[k]}}), \psi \neq 0} \frac{Q_{\mathbf{A}^{[k]}}(\psi)}{\|\psi\|_{\mathsf{L}^p}^2},$$

where $\mathbf{A}^{[k]}(x, y) = \left(0, \frac{x^{k+1}}{k+1}\right)$, (i.e. $\mathbf{B}^{[k]}(x, y) = x^k$). Here

$$Q_{\mathbf{A}^{[k]}}(\psi) = \int_{\mathbb{R}^2} |(-i\nabla + \mathbf{A}^{[k]})\psi|^2 \, \mathrm{d}\mathbf{x},$$

with domain

$$\mathrm{Dom}(Q_{\mathbf{A}^{[k]}}) = \left\{ \psi \in \mathsf{L}^2(\mathbb{R}^2) \, : \, (-i\nabla + \mathbf{A}^{[k]})\psi \in \mathsf{L}^2(\mathbb{R}^2) \right\}.$$

It is not obvious that there is a minimizer for the global problem. Indeed, for $p = 2$, $k \geq 1$ there is not.

Theorem (SF-Nicolas Raymond)

Assume that $p \geq 2$, that $\mathbf{A}$ is smooth on $\overline{\Omega}$, that $\mathbf{B} = \nabla \times \mathbf{A} > 0$ does not vanish on $\overline{\Omega}$ and that its minimum b_0 is attained in Ω. Then there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,

$$(1 - Ch^{\frac{1}{8}})\lambda^{[0]}(p) b_0^{\frac{2}{p}} h^{2-\frac{2}{p}} \leq \lambda(\Omega, \mathbf{A}, p, h) \leq (1 + Ch^{1/2})\lambda^{[0]}(p) b_0^{\frac{2}{p}} h^{2-\frac{2}{p}}.$$

Moreover, if the minimum is attained on the boundary, the lower bound is still valid.

Previous results by Di Cosmo and van Schaftingen (preprint - JDE), without error bound.

Semiclassical decay

Theorem (SF-Nicolas Raymond)

Let $p > 2$, $\rho \in \left(0, \frac{1}{2}\right)$, $\varepsilon > 0$. Let the assumptions be as in the previous theorem and assume furthermore that the minimum of the magnetic field is unique and attained at $\mathbf{x}_0 \in \Omega$.
Then there exist $C_1, C_2 > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$ and all non-linear minimizers ψ, we have

$$\|\psi\|_{L^\infty(\complement D(\mathbf{x}_0, 2\varepsilon))} \leq C e^{-Ch^{-\rho}} \|\psi\|_{L^\infty(\Omega)},$$

where $D(\mathbf{x}, R)$ denotes the open ball of center $\mathbf{x}$ and radius $R > 0$.

Vanishing magnetic fields

Theorem (SF-Nicolas Raymond)

Let $p > 2$. Let us assume that $\mathbf{A}$ is smooth on $\overline{\Omega}$, that

$$\Gamma = \{\mathbf{x} \in \overline{\Omega} \,:\, \mathbf{B}(\mathbf{x}) = 0\},$$

satisfies that $\Gamma \subset \Omega$ is a smooth, simple and closed curve, and that $\mathbf{B}$ vanishes non-degenerately along Γ in the sense that

$$\nabla \mathbf{B}(\mathbf{x}) \neq 0, \qquad \textit{for all } \mathbf{x} \in \Gamma.$$

Assuming that $\mathbf{B}$ is positive inside Γ and negative outside, we denote by $\gamma_0 > 0$ the minimum of the normal derivative of $\mathbf{B}$ with respect to Γ. Then there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,

$$(1 - Ch^{\frac{1}{33}})\lambda^{[1]}(p)\gamma_0^{\frac{4}{3p}} h^{2-\frac{4}{3p}} \leq \lambda(\Omega, \mathbf{A}, p, h) \leq (1 + Ch^{\frac{1}{3}})\lambda^{[1]}(p)\gamma_0^{\frac{4}{3p}} h^{2-\frac{4}{3p}}.$$

Difficulties compared to linear case, $p = 2$

- Existence (and decay) of minimizers of model problems. (Desirable for getting good upper bounds).
- Localizations : If $\sum \chi_j^2 = 1$, what can we then say about the comparison between $\sum \|\chi_j \phi\|_p^2$ and $\|\phi\|_p^2 = \| \sum \chi_j^2 \phi\|_p^2$.

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Lower bounds ($b_0 > 0$)

Let $0 < \rho \le \alpha$ (so $h^{\alpha} \le h^{\rho}$). Partition of unity :

$$\sum_j \chi_j^2 = 1, \qquad \operatorname{diam}\operatorname{supp}\chi_j \le Ch^{\rho}, \qquad \sum_j |\nabla \chi_j|^2 \le Ch^{-2\alpha}.$$

$$\begin{aligned}
\mathcal{Q}_{h,\mathbf{A}}(\psi) &\ge \sum_j \left(\mathcal{Q}_{h,\mathbf{A}}(\chi_j\psi) - Ch^{2-2\alpha}\|\chi_j\psi\|_2^2\right) \quad \text{(IMS)} \\
&\ge \sum (1 - Ch^{1-2\alpha})\mathcal{Q}_{h,\mathbf{A}}(\chi_j\psi) \\
&\ge \sum (1 - Ch^{1-2\alpha} - Ch^{2\rho-\frac{1}{2}})\mathcal{Q}_{h,b_j\mathbf{A}^{[0]}}(e^{i\phi_j/h}\chi_j\psi) \quad (\text{Approx cst } \mathbf{B}) \\
&\ge \sum_j (1 - Ch^{1-2\alpha} - Ch^{2\rho-\frac{1}{2}}) b_j^{2/p} h^{2-2/p} \lambda^{[0]}(p)\|\chi_j\psi\|_p^2
\end{aligned}$$

So we would like a bound of the form $\sum \|\chi_j\psi\|_p^2 \ge \|\psi\|_p^2$.

$\sum \|\chi_j \psi\|_p^2 \geq \|\psi\|_p^2$?

Since $2/p \leq 1$,

$$\sum \|\chi_j\psi\|_p^2 = \sum \left(\int |\chi_j \psi(x)|^p dx \right)^{2/p}$$
$$= \left(\sum \int |\chi_j \psi(x)|^p dx \right)^{2/p} \sum \left(\frac{\int |\chi_j \psi(x)|^p dx}{\sum \int |\chi_j \psi(x)|^p dx} \right)^{2/p}$$
$$\geq \left(\sum \int |\chi_j \psi(x)|^p dx \right)^{2/p}$$

So we need to compare

$$\sum \int |\chi_j \psi(x)|^p dx \overset{?}{\approx} \int |\psi|^p dx = \sum \int \chi_j^2 |\psi|^p dx$$

Clearly, $(\chi_j(x) \leq 1)$,

$$\sum \int |\chi_j \psi(x)|^p dx \leq \sum \int \chi_j^2 |\psi|^p dx.$$

Sliding grids

Standard partition of unity

$$\chi_j = \chi(\cdot - j\ell), \qquad j \in \mathbb{Z}^2, \ell = 2h^\rho + h^\alpha, \alpha \geq \rho$$
$$\chi = 1 \text{ on } \{\|x\|_\infty \leq h^\rho\}, \qquad \operatorname{supp}\chi \subset \{\|x\|_\infty \leq h^\rho + h^\alpha\}.$$

Translates : $\chi_{j,\tau} = \chi(\cdot - j\ell - \tau)$, $\tau \in [0,\ell)^2$. Then

$$\int_{[0,\ell)^2} \sum_j \chi(x - j\ell - \tau) d\tau = \int_{\mathbb{R}^2} \chi \geq (2h^\rho)^2.$$

$$\frac{1}{\ell^2} \int_{[0,\ell)^2} \left(\sum \int |\chi_{j,\tau}(x)\psi(x)|^p dx \right) d\tau \geq \int |\psi(x)|^p dx \times \frac{1}{\ell^2}(2h^\rho)^2$$
$$\geq \|\psi(x)\|_p^p \times (1 + \mathcal{O}(h^{\alpha-\rho})).$$

In particular, there exists τ_0 (depending on ψ and h) such that

$$\sum \int |\chi_{j,\tau_0}(x)\psi(x)|^p dx \geq \|\psi(x)\|_p^p \times (1 + \mathcal{O}(h^{\alpha-\rho})).$$

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Wrapping it up

$$
\begin{aligned}
\mathcal{Q}_{h,\mathbf{A}}(\psi) &\geq \sum_j (1 - Ch^{1-2\alpha} - Ch^{2\rho-\frac{1}{2}}) b_j^{2/p} h^{2-2/p} \lambda^{[0]}(p) \|\chi_{j,\tau}\psi\|_p^2 \\
&\geq (1 - Ch^{1-2\alpha} - Ch^{2\rho-\frac{1}{2}}) b_0^{2/p} h^{2-2/p} \lambda^{[0]}(p) \left(\sum \|\chi_{j,\tau}\psi\|_p^p \right)^{2/p} \\
&\geq (1 - Ch^{1-2\alpha} - Ch^{2\rho-\frac{1}{2}}) b_0^{2/p} h^{2-2/p} \lambda^{[0]}(p) \times \\
&\qquad \left((1 + \mathcal{O}(h^{\alpha-\rho})) \|\psi\|_p^p \right)^{2/p}.
\end{aligned}
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Parameters : $\rho = \frac{5}{16}$, $\alpha = \frac{7}{16}$.

Existence of minimizers of model problems

Case $k = 1$.

$$\lambda^{[1]}(p) = \lambda(\mathbb{R}^2, \mathbf{A}^{[1]}, p, 1) = \inf_{\psi \in \mathrm{Dom}(Q_{\mathbf{A}^{[1]}}), \psi \neq 0} \frac{Q_{\mathbf{A}^{[1]}}(\psi)}{\|\psi\|_{\mathsf{L}^p}^2},$$

where $\mathbf{A}^{[1]}(x, y) = \left(0, \frac{x^2}{2}\right)$, (i.e. $\mathbf{B}^{[1]}(x, y) = x$). Here

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Theorem (SF-Raymond, Esteban-Lions)

For $p > 2$, $\lambda^{[1]}(p)$ is a minimum. I.e. there exists a minimizer.

Concentration-compactness

Let u_n be a minimizing sequence, $\|u_n\|_p = 1$,

$$Q_{\mathbf{A}^{[1]}}(u_n) \searrow \lambda^{[1]}(p).$$

Define $\mu_n = |u_n|^2 + |(-i\nabla + \mathbf{A}^{[1]})u_n|^2$. Three possibilities for the measures μ_n (up to extraction of subsequences) [Lions].

1. Vanishing. $\forall R > 0$, we have

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathbb{R}^2} \mu_n(D(\mathbf{x}, R)) = 0.$$

2. Dichotomy. $\exists \beta \in (0,\mu), \forall \varepsilon > 0, \exists R_1 > 0, \exists R_n \to +\infty, (y_n)_{n\in\mathbb{N}}$,

$$|\mu_n(D(y_n, R_1)) - \beta| \leq \varepsilon, \quad |\mu_n(\complement D(y_n, R_n)) - (\mu - \beta)| \leq \varepsilon.$$

3. Tightness. There exists $(\mathbf{x}_n)_{n\geq 0}$ such that

$$\forall \varepsilon > 0, \quad \exists R > 0, \quad \forall n \geq 1, \quad \mu_n(\mathbb{R}^2 \setminus D(\mathbf{x}_n, R)) \leq \varepsilon.$$

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The case 3. Tightness

There exists $(\mathbf{x}_n)_{n\geq 0}$, $\mathbf{x}_n = (x_n, y_n)$, such that

$$\forall \varepsilon > 0, \quad \exists R > 0, \quad \forall n \geq 1, \quad \mu_n(\mathbb{R}^2 \setminus D(\mathbf{x}_n, R)) \leq \varepsilon.$$

OBS. $\mu_n(\mathbb{R}^2) \to \mu > 0$ [Since $\mu = 0 \Rightarrow u_n \to 0$ in $H^1_{\mathbf{A}}(\mathbb{R}^2)$ which implies $u_n \to 0$ in L^p].

Center the balls at 0 : $\hat{u}_n = u_n(\cdot + \mathbf{x}_n)$, $\mathbf{A}_n(\mathbf{x}) = \mathbf{A}(\mathbf{x} - \mathbf{x}_n) = (0, \frac{(x-x_n)^2}{2})$, Then

$$\int_{D(0,R)} |D_x \hat{u}_n|^2 + \left| \left(D_y + \frac{(x-x_n)^2}{2} \right) \hat{u}_n \right|^2 \mathrm{d}\mathbf{x} \leq C,$$

Since $\mathbf{B}_n = (x - x_n)$, this implies that x_n remains bounded.
(From a spectral analysis of this quadratic form we get $x_n \to \infty$ $\Rightarrow \|u_n\|_2 \to 0$. But since $\|u_n\|_{H^1}$ is bounded, this would imply $u_n \to 0$ in L^p, contradicting the normalization.)

Banach-Alaoglu : $\hat{u}_n \rightharpoonup \hat{u}_*$ weakly in $H^1_{\mathbf{A}}(\mathbb{R}^2)$.

$$\int_{D(0,R)} |D_x \hat{u}_n|^2 + \left| \left(D_y + \frac{(x - x_n)^2}{2} \right) \hat{u}_n \right|^2 \mathrm{d}\mathbf{x} \leq C,$$

Since x_n is bounded, this gives $\hat{u}_n$ bounded in $H^1(D(0,R))$. So $\hat{u}_n \to \hat{u}_*$ strongly in $L^2(D(0,R))$, so by tightness $\hat{u}_n \to \hat{u}_*$ strongly in $L^2(\mathbb{R}^2)$. So by Sobolev, $\hat{u}_n \to \hat{u}_*$ strongly in $L^p(\mathbb{R}^2)$, i.e. $\|\hat{u}_*\|_p = 1$.
So by Fatou,

$$\begin{aligned} \liminf Q_{\mathbf{A}^{[1]}}(u_n) &= \liminf \int |D_x \hat{u}_n|^2 + \left| \left(D_y + \frac{(x - x_n)^2}{2} \right) \hat{u}_n \right|^2 \mathrm{d}\mathbf{x} \\ &\geq \int |D_x \hat{u}_*|^2 + \left| \left(D_y + \frac{(x - x_*)^2}{2} \right) \hat{u}_* \right|^2 \mathrm{d}\mathbf{x}. \end{aligned}$$

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Banach-Alaoglu : $\hat{u}_n \rightharpoonup \hat{u}_*$ weakly in $H^1_{\mathbf{A}}(\mathbb{R}^2)$.

$$\int_{D(0,R)} |D_x \hat{u}_n|^2 + \left| \left(D_y + \frac{(x - x_n)^2}{2} \right) \hat{u}_n \right|^2 \mathrm{d}\mathbf{x} \leq C,$$

Since x_n is bounded, this gives $\hat{u}_n$ bounded in $H^1(D(0,R))$. So $\hat{u}_n \to \hat{u}_*$ strongly in $L^2(D(0,R))$, so by tightness $\hat{u}_n \to \hat{u}_*$ strongly in $L^2(\mathbb{R}^2)$. So by Sobolev, $\hat{u}_n \to \hat{u}_*$ strongly in $L^p(\mathbb{R}^2)$, i.e. $\|\hat{u}_*\|_p = 1$.
So by Fatou,

$$\begin{aligned} \liminf Q_{\mathbf{A}^{[1]}}(u_n) &= \liminf \int |D_x \hat{u}_n|^2 + \left| \left(D_y + \frac{(x - x_n)^2}{2} \right) \hat{u}_n \right|^2 \mathrm{d}\mathbf{x} \\ &\geq \int |D_x \hat{u}_*|^2 + \left| \left(D_y + \frac{(x - x_*)^2}{2} \right) \hat{u}_* \right|^2 \mathrm{d}\mathbf{x}. \end{aligned}$$