branching diffusions with immigration: time-discrete observation, a reconstruction algorithm for particle trajectories, and estimation of the diffusion coefficient

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our aims

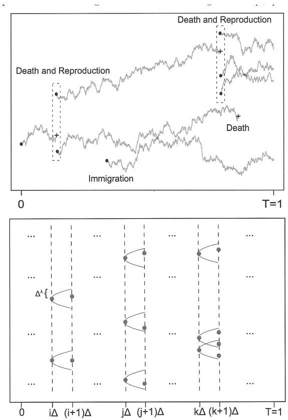
branching diffusion with immigration (BDI), ergodic setting,
time-continuous observation $\leftrightarrow$ time-discrete observation at small step size Δ:

in continuous time:

- independent diffusion paths
- position-dependent branching rate
- random displacement of offspring
- immigration at constant rate

in discrete time:

- pairs of successive configurations
- no information on particle identities
- seemingly 'identifiable' pairs – and others
- $\rightarrow$ reconstruction algorithm ??
- $\rightarrow$ regression schemes for estimation ??



BDI process, ergodicity, invariant measure

single particle space $(E, \mathcal{E})$: $E = \mathbb{R}^d$ with Borel σ-field

configuration space $(S, \mathcal{S})$: $S = \bigcup_{\ell \in \mathbb{N}_0} E^\ell$ with Borel σ-field, $E^0 = \{\delta\}$ void conf.

$x = (x_1, \ldots, x_\ell)$ elements of S, $\ell : S \to \mathbb{N}_0$ length of a configuration,
$x \in S$, $A \in \mathcal{E}$: $x(A) = \sum_{i=1}^{\ell(x)} \epsilon_{x_i}(A)$ number of particles visiting A,
$f : E \to \mathbb{R}$ a function, $x \in S$: $\overline{f}(x) = \sum_{i=1}^{\ell(x)} f(x_i)$ with convention $\overline{f}(\delta) := 0$

particles travel on independent diffusion paths $d\xi_s = b(\xi_s)ds + \sigma(\xi_s)dW_s$,
$a = \sigma\sigma^\top$, assume: drift $\mathcal{C}_b^1(E)$, diffusion coefficient $\mathcal{C}_b^2(E)$

branching at position-dependent rate $\kappa(\cdot) \in \mathcal{C}_b(E)$

when branching happens in position $y \in E$: number k of descendants and locations $y + v_1, \ldots, y + v_k$ for offspring selected by Markov kernel

$$K_1(y, dk)K_2((y, k), dv_1, \ldots, dv_k) = p_k(y) \prod_{j=1}^{k} Q^{\mathrm{r}}(dv_j) ,$$

finite reproduction means: $\rho(\cdot) = \sum_{k \geq 1} k p_k(\cdot)$ in $\mathcal{C}_b(E)$

immigration: PPP on $(0, \infty) \times E$ with intensity $c\,ds\, Q^{\mathrm{i}}(dy)$

with these ingredients:
construct the BDI process as S-valued strong Markov process $(\eta_t)_{t\geq 0}$

- infinite lifetime, no accumulation of jumps in finite time, jumps (branching or immigration) arriving at rate $(c+\overline{\kappa}(\eta_s))ds$
- sequence $T_n \uparrow \infty$ of stopping times such that on $[[T_n, T_{n+1}[[$, ℓ-particle configurations travel on diffusion paths

subprocesses $(\eta^{\mathfrak{r}}_{s+h})_{h\geq 0}$ of all direct descendants of one ancestor at time s are branching diffusions without immigration, occupation time kernel

$$H(y,g) = E_y\left(\int_0^\infty \overline{g}(\eta^{\mathfrak{r}}_h)\, dh\right) \ , \quad y\in E, g\in \mathcal{E}^+$$

for which there exists a jump diffusion $\widetilde{\xi}$ such that

$$H(y,g) = E_y\left(\int_0^\infty g(\widetilde{\xi}_t)\, e^{-\int_0^t [\kappa(1-\rho)](\widetilde{\xi}_s)\, ds}\, dt\right)$$

specified through its generator (writing $\mathcal{L}$ for the generator of ξ)

$$\widetilde{\mathcal{L}}g(y) = \mathcal{L}g(y) + [\kappa\rho](y)\int_E Q^{\mathfrak{r}}(dv)[g(y+v)-g(y)]$$

proposition 1: the BDI process $(\eta_t)_{t\geq 0}$ is positive Harris recurrent if

(∗) $y \to H(y,1)$ is finite and belongs to $L^1(Q^{\mathrm{i}})$

then: void configuration δ is a recurrent atom, invariant measure on S:

$$\mu(F) = E_\delta\left(\int_0^R 1_F(\eta_s)\,ds\right) \quad , \quad F \in \mathcal{S}$$

with R the time of first return to δ, invariant occupation measure on E:

$$\overline{\mu}(A) = E_\delta\left(\int_0^R \eta_s(A)\,ds\right) \quad , \quad A \in \mathcal{E}\ ;$$

by (∗), $\overline{\mu}$ is a finite measure on E given by (up to constant multiples)

$$\overline{\mu} = Q^{\mathrm{i}} H \quad , \quad \overline{\mu}(E) = E_{Q^{\mathrm{i}}}\left(\int_0^\infty e^{-\int_0^t [\kappa(1-\rho)](\widetilde{\xi}_s)\,ds}\,dt\right) < \infty$$

(case $Q^{\mathrm{r}} = \epsilon_0$: cf. Ikeda, Nagasawa and Watanabe 1969, Nagasawa 1977, Löcherbach 2004, H-L 2005, ... our case: jump laws Q^{r} allow for continuous Lebesgue density of μ on S, Hammer 2012)

more on the invariant measure

associate to the $[\kappa(1-\rho)]$-potential kernel $H(y,g)$ expectation semigroup

$$M_t(y,g) = E_y\left(g(\widetilde{\xi}_t)\, e^{-\int_0^t [\kappa(1-\rho)](\widetilde{\xi}_s)\, ds} \right) \;, \quad t \geq 0$$

(Ikeda, Nagasawa and Watanabe 1969), write $|||M_t||| = \sup_{y\in E} M_t(y,E)$.

theorem 1: can construct $\alpha^\top$ in $\mathcal{C}_b(E)$ and a semimartingale of finite jump intensity $\widetilde{\xi}^\top$ with generator $\widetilde{\mathcal{L}}^\top$ and semigroup

$$M_t^\top(y,g) = E_y\left(g(\widetilde{\xi}_t^\top)\, e^{-\int_0^t \alpha^\top(\widetilde{\xi}_s^\top)\, ds} \right) \;, \quad t \geq 0$$

such that duality of Feller semigroups holds:

$$\langle (\widetilde{\mathcal{L}} - [\kappa(1-\rho)])f \,,\, g \rangle = \langle f \,,\, (\widetilde{\mathcal{L}}^\top - \alpha^\top) g \rangle \;, \quad f, g \in \mathcal{C}_c^\infty(E)$$

$$\langle M_t f, g \rangle = \langle f, M_t^\top g \rangle \;, \quad t \geq 0$$

$$|||M_t||| \leq e^{t\, \|[\kappa(1-\rho)]\|_\infty} \quad , \quad |||M_t^\top||| \leq e^{t\, \|\alpha^\top\|_\infty}$$

(proofs using strongly continuous semigroups, approximation results, bounded operators on $L^1(E)$, bounded perturbation – no flows of diffeomorphisms) formally by prop. 1 and thm. 1, when q^{i} is a Lebesgue density for Q^{i}, write

$$\overline{\mu} f = Q^{\mathrm{i}} H\, f = \langle\, q^{\mathrm{i}}, Hf \,\rangle = \langle\, H^{\top} q^{\mathrm{i}}, f \,\rangle \;\leq\; \infty$$

where $H^{\top}(y,g)$ is the resolvent of the semigroup $(M_t^{\top})_{t\geq 0}$, thus

$$y \longrightarrow H^{\top}(y, q^{\mathrm{i}}) = \int_0^{\infty} dt\, M_t^{\top}(y, q^{\mathrm{i}}) = E_y \left(\int_0^{\infty} q^{\mathrm{i}}(\widetilde{\xi}_t^{\top})\, e^{-\int_0^t \alpha^{\top}(\widetilde{\xi}_s^{\top})\, ds}\, dt \right)$$

should be a Lebesgue density for $\overline{\mu}$ on E

theorem 2: assume $(*)$ and

$$(**) \qquad \limsup_{t\to\infty} \frac{1}{t} \log |||M_t||| \;<\; 0 \quad , \quad \limsup_{t\to\infty} \frac{1}{t} \log |||M_t^{\top}||| \;<\; 0$$

a) if $q^{\mathrm{i}} \in \mathcal{C}_0(E)$, then the invariant occupation measure $\overline{\mu}$ on E admits a Lebesgue density $\overline{\gamma}(\cdot) = H^{\top}(\cdot, q^{\mathrm{i}})$ which is $\mathcal{C}_0(E)$

b) for $q \in \mathbb{N}$: if condition (Mq): $y \to \sum_k k^q p_k(y)$ is $\mathcal{C}_b(E)$ holds, then

$$\mu(\ell^q) = \int_S \ell^q(x)\, \mu(dx) = \sum_{\ell} \ell^q \mu(E^{\ell}) \;<\; \infty$$

the reconstruction algorithm (RA)

from now on: for the semigroup $(P_t^\kappa(y,dz))_{t\geq 0}$ on $E=\mathbb{R}^d$ corresponding to one-particle motion ξ killed at rate $\kappa(\cdot)$, we assume heat kernel bounds

$$(HKB) \qquad p_t^\kappa(y,z) \;\leq\; C\,t^{-\frac{d}{2}}\,e^{-\frac{1}{2}\frac{|z-y|^2}{C\,t}}\ , \quad 0<t\leq t_0\ ,\ y,z\in E$$

definition 1: call a two-particle configuration $(x,x')\in E\times E$ ε-wellspread if $\min_{1\leq j\leq d}|x_j-x_j'|\;\geq\;\varepsilon$; call a configuration $x\in S$ ε-wellspread if arbitrary two-particle subconfigurations of x are ε-wellspread

write N_ε for the set of all configurations in S which are not ε-wellspread

theorem 3: under (HKB) and $(M3)$, we have

$$\mu(N_\varepsilon)\;\leq\;O(\varepsilon)\quad \text{as } \varepsilon\downarrow 0.$$

proof: resolvant calculations for multi-particle motions under killing rate $\kappa(\cdot)$ establish $\mu(N_\varepsilon\cap E^\ell)\leq D\,\varepsilon\,\ell^3\mu(E^\ell)$ for $\ell\geq 2$

when we observe discretely in time: recording pairs of successive observations $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$ only, any information on individual particle trajectories between times $i\Delta$ and $(i+1)\Delta$ is lost ... $\hookrightarrow$ problem of particle identification !!!

definition 2: call $(x,y) \in S \times S$ (Δ,λ)-identifiable if $(\Delta > 0,\ 0 < \lambda < \frac{1}{2})$

- $\ell(x) = \ell(y) = \ell$ for some $\ell \in \mathbb{N}$
- $x = (x_1, \ldots, x_\ell)$ is $4\Delta^\lambda$-wellspread, $y = (y_1, \ldots, y_\ell)$ is $2\Delta^\lambda$-wellspread
- exists permutation π of the ℓ particles (necessarily unique) such that

$$\max_{1 \le j \le d} |y_{\pi(m),j} - x_{m,j}| < \Delta^\lambda \quad , \quad 1 \le m \le \ell$$

write $\mathtt{ID}(\Delta,\lambda)$ for the subset of (Δ,λ)-identifiable pairs in $S \times S$, call pairs $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$ (Δ,λ)-identifiable if $(\eta_{i\Delta}, \eta_{(i+1)\Delta}) \in \mathtt{ID}(\Delta,\lambda)$ holds

reconstruction algorithm: for (Δ,λ)-identifiable pairs $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$, writing $x := \eta_{i\Delta}$, $y := \eta_{(i+1)\Delta}$, ℓ and π as in definition 2, we decide to view

$$(RA) \qquad \left\{ \begin{array}{l} y_{\pi(m)} \text{ as the position at time } (i+1)\Delta \text{ of the particle} \\ \text{which was in position } x_m \text{ at time } i\Delta, \text{ for } 1 \le m \le \ell \end{array} \right.$$

(of course, this decision may be wrong ...)

theorem 4: for $(\eta_t)_{t\geq 0}$ stationary, $0<\lambda<\frac{1}{2}$ fixed, as $\Delta\downarrow 0$:

$$Q_\mu\left((\eta_{i\Delta},\eta_{(i+1)\Delta}) \notin \mathtt{ID}(\Delta,\lambda) \text{ or } \text{(RA) decision incorrect} \right) \;\leq\; O(\Delta^\lambda)$$

proof: define 'good = correctly identifiable' path segments

$$\eta_{[i\Delta,(i+1)\Delta]} := (\eta_s)_{i\Delta\leq s\leq(i+1)\Delta}$$

as elements of the cadlag path space $D([i\Delta,(i+1)\Delta],S)$ such that

$$\{\eta_{[i\Delta,(i+1)\Delta]}\text{'good'}\} \;\subset\; \{(\eta_{i\Delta},\eta_{(i+1)\Delta})\in \mathtt{ID}(\Delta,\lambda) \text{ and } \text{(RA) decision is correct}\}$$

we prove under assumptions (HKB) and $(M1)$ that

$$Q_\mu\left(\eta_{[i\Delta,(i+1)\Delta]} \text{ is } (\Delta,\lambda)\text{-good}\right) \;\geq\; 1-O(\Delta^\lambda)$$

as $\Delta\downarrow 0$, with leading contribution on l.h.s. (cf. thm. 3 and def. 2)

$$Q_\mu\left(\eta_{i\Delta} \text{ is } 4\Delta^\lambda\text{-wellspread}\right) = 1-\mu(N_{4\Delta^\lambda}) \;\geq\; 1-O(\Delta^\lambda)$$

which explains the rate in theorem 4; as a by-product:

$$(+) \qquad Q_\mu\left((\eta_{i\Delta},\eta_{(i+1)\Delta})\in \mathtt{ID}(\Delta,\lambda)\,,\; \text{(RA) decision incorrect}\right) \;\leq\; O(\Delta)$$

filling regression schemes for $\sigma^2(\cdot)$ (dim $d = 1$)

$E := \mathbb{R}$, let $\mathtt{CI}(\Delta, \lambda)$ denote the set of '(Δ,λ)-good' path segments:

$$\{\, \eta_{[m\Delta,(m+1)\Delta]} \in \mathtt{CI}(\Delta,\lambda) \,\} \subset \{\, (\eta_{m\Delta}, \eta_{(m+1)\Delta}) \in \mathtt{ID}(\Delta,\lambda) \,\}$$

consider any interval A such that $\inf_{x \in A} \overline{\gamma}(x) > 0$, w.l.o.g. $A := [0,1]$

subdivide A into n cells A_j of equal length, by ergodicity

(since $\lim_{i\to\infty} \frac{1}{i} \sum_{m=1}^{i} 1_{\{\eta_{m\Delta}(A_j)\geq 1\}} 1_{\{\, \eta_{[m\Delta,(m+1)\Delta]} \in \mathtt{CI}(\Delta,\lambda)\,\}}$ exists in $(0,\infty)$)

associate to every cell A_j a stopping time τ_j with finite mean, $1 \leq j \leq n$:

$$\tau_j \;:=\; \min\left\{ m \in \mathbb{N}_0 : \eta_{m\Delta}(A_j) \geq 1 \text{ and } (\eta_{m\Delta}, \eta_{(m+1)\Delta}) \in \mathtt{ID}(\Delta,\lambda) \right\}$$

take some m_j such that particle m_j in configuration $\eta_{\tau_j\Delta}$ visits A_j, then put

$$(RS) \quad \mathcal{X}_j := (\eta_{\tau_j\Delta})_{m_j} \in A_j \;,\; \mathcal{Y}_j := \left(\frac{(\eta_{(\tau_j+1)\Delta})_{\pi(m_j)} - (\eta_{\tau_j\Delta})_{m_j}}{\sqrt{\Delta}} \right)^2 ,\; 1 \leq j \leq n$$

with π prescribed by the reconstruction algorithm (RA) (we are allowed to fill several cells at the same time) theorem 5: assume $d = 1$, (HKB), (M3), ..., then the regression scheme (RS)

$$(\mathcal{X}_j \, , \, \mathcal{Y}_j) \; , \; 1 \le j \le n$$

has the following properties as $n \to \infty$ and $\Delta \downarrow 0$:

a) the $\mathcal{X}_j$, $1 \le j \le n$, are approximately aequidistant

b) on an exceptional event of small probability $\le O(n\Delta)$,
some $\mathcal{Y}_j$'s in (RS) may not correspond to underlying one-particle motions; but (RA) always provides a trivial deterministic bound $|\mathcal{Y}_j| \le \Delta^{2\lambda-1}$, $1 \le j \le n$

c) on an event of large probability $\ge 1 - O(n\Delta)$, the scheme (RS) satisfies

$$\mathcal{Y}_j \;=\; \sigma^2(\mathcal{X}_j)(1+U_j) + R_j \; , \quad 1 \le j \le n$$

where for iid BM's W_j, $1 \le j \le n$, iid pairs (U_j, R_j) and some global cst

$$U_j = 2\int_0^1 W_j dW_j = [W_j(1)^2 - 1] \quad , \quad E(R_j^2) \le C\,\Delta$$

(proof: recall $(\eta_{\tau_j \Delta}, \eta_{(\tau_j+1)\Delta}) \in \mathtt{ID}(\Delta, \lambda)$ for all $1 \le j \le n$ by construction;
c) $\leftrightarrow$ event that (RA) decides correctly, thus all $\mathcal{Y}_j$ are taken out of some true underlying one-particle motion; b) $\leftrightarrow$ event $(+)$ in theorem 4)

estimation of the diffusion coefficient at points $a \in \mathrm{int}(A)$, $A = [0,1]$: for n large and Δ small, consider regression schemes (RS)

unknown $\sigma^2(\cdot)$, assume $\sigma^2 \in \mathcal{H}^\beta$, the Hölder class of smoothness $\beta > 2$

consider as an example kernel estimators (other examples could be local polynomial estimators, cf. Tsybakov 2009 section 1.6)

kernel K of order $\lfloor \beta \rfloor$, bandwidth $h = n^{-\frac{1}{2\beta+1}}$, $K_h = \frac{1}{h} K(\frac{\cdot}{h})$, estimator

$$\widehat{\sigma^2_{n,\Delta}}(a) \ := \ \frac{1}{n} \sum_{j=1}^{n} \mathcal{Y}_j \, K_h(\mathcal{X}_j - a)$$

(since $\mathcal{X}_j$, $1 \le j \le n$, approximately aequidistant: $\frac{1}{n} \sum_{j=1}^{n} K_h(\mathcal{X}_j - a) \approx 1$)

quadratic risk when estimating from (RS)

$$\sup_{\sigma^2 \in \mathcal{H}^\beta} E_{\sigma^2} \left(\left| \widehat{\sigma^2_{n,\Delta}}(a) - \sigma^2(a) \right|^2 \right) \ \le \ O\left(n \, \Delta^{2\lambda} \right) + O\left(n^{-\frac{2\beta}{2\beta+1}} \right)$$

arises as the sum of two terms:

- on the exceptional set of probability $O(n\Delta)$: all $|\mathcal{Y}_j| \le \Delta^{2\lambda - 1}$
- usual nonparametric squared risk on event of probability $1 - O(n\Delta)$

both terms will be of same order

$$O\left(n^{-\frac{2\beta}{2\beta+1}}\right) = O\left(\frac{1}{n\,h_n}\right) = O\left(n\,\Delta^{2\lambda}\right)$$

if we take $\Delta = \Delta_n$ such that $\Delta^{2\lambda} = \frac{1}{n^2 h_n} = n^{-\frac{4\beta+1}{2\beta+1}}$: we thus arrive at

corollary: estimating unknown $\sigma^2 \in \mathcal{H}^\beta$, $\beta > 2$ fixed, from time-discrete observation of the BDI process at step size Δ

$$\Delta = \Delta_n \text{ such that } \Delta^\lambda = \frac{1}{n\sqrt{h_n}} = \frac{1}{n}\, n^{+\frac{1/2}{2\beta+1}}$$

using reconstruction algorithm (RA) to fill a regression scheme (RS) with n cells, we have asymptotically as $n \to \infty$

$$\sup_{\sigma^2 \in \mathcal{H}^\beta} E_{\sigma^2}\left(\left|\widehat{\sigma^2_{n,\Delta}}(a) - \sigma^2(a)\right|^2\right) \leq O\left(n^{-\frac{2\beta}{2\beta+1}}\right)$$

i.e. we attain the nonparametric rate known to be optimal (Tsybakov 2009 sect. 2.5) for squared risk in standard regression schemes for unknown $f \in \mathcal{H}^\beta$

$$(\mathcal{U}_j, \mathcal{V}_j)\ ,\ \mathcal{V}_j = f(\mathcal{U}_j) + \epsilon_j\ ,\ 1 \leq j \leq n$$

with iid errors ϵ_j and equispaced deterministic $\mathcal{U}_j$

–THE END–

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