Periodic Magnetic Schrödinger Operator in Superconductivity

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(FK2D) S. Fournais, A. Kachmar. *Advances in Mathematics.* (2010)

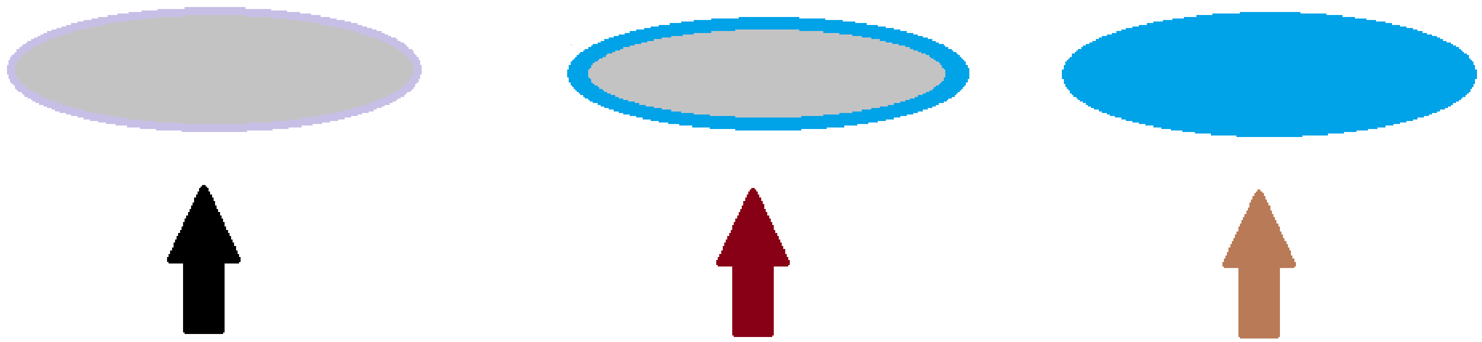
(FK3D) S. Fournais, A. Kachmar. *Communications in PDE.* (2013)

(Kac) A. Kachmar. *SIAM J. Math. Anal.* (2014)

(HK) B. Helffer, A. Kachmar. *arXiv:*1503.08529

Superconductivity and Magnetic Fields

As the intensity of the magnetic field gradually **decreases** and crosses the critical values H_{C_3} and H_{C_2}, superconductivity is restored first in the **surface** then in the **bulk**. Note: $H_{C_3} > H_{C_2}$.



Spectral Theory and Superconductivity

The operator $P_B = -(\nabla - iB\mathbf{A}_0)^2$ appears in

(StJdeG) St. James, P.G. de Gennes.

(FH) S. Fournais, B. Helffer. *Progress in NDE and their applications.* (Vol. 77)

to compute the third critical field H_{C_3}.

Superconductivity is **lost** in samples subject to a magnetic field of intensity $B > H_{C_3}$

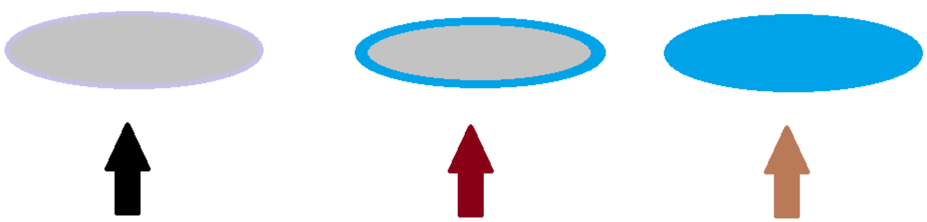
Spectral Theory and Superconductivity

$$B > 0\,, \quad \mathbf{A}_0(x_1,x_2) = \frac{1}{2}(-x_2,x_1)\,, \quad P_B = -(\nabla - iB\mathbf{A}_0)^2\,.$$

- P_B in $\mathbb{R}^2$ (spectrum=Landau levels=$\{1B, 3B, etc\}$)
- P_B in $\mathbb{R}^2_+$ (spectrum=$[B\Theta_0, \infty)$ - $\Theta_0 \approx 0.59$)
- P_B in a domain $\Omega \subset \mathbb{R}^2$ with **Dirichlet** or **Neumann** condition
- Application: $H_{C_2} \approx \kappa$; $H_{C_3} \approx \kappa/\Theta_0$

The Critical Field H_{C_2}
Superconductivity is Back.. in the **Bulk**:

Abrikosov: There is a critical field $H_{C_2} \approx \kappa$ such that, near H_{C_2}, superconductivity is uniformly distributed in the bulk of the sample. Between H_{C_2} and H_{C_3} superconductivity **disappears in the bulk** and **appears in the surface**.



Magnetic Periodic Condition (Mag Per Cond):

- **The magnetic potential:** $\mathbf{A}_0(x_1, x_2) = \frac{1}{2}(-x_2, x_1)$

- **The lattice's cell:** $Q_R = (-R, R) \times (-R, R) \subset \mathbb{R}^2$

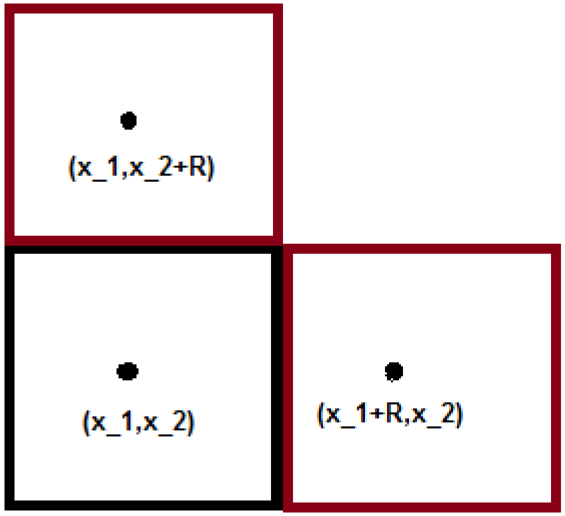
- $\mathbf{A}_0(x_1 + R, x_2 + R) = \mathbf{A}_0(x_1, x_2) + \frac{R}{2}\nabla(-x_1 + x_2)$

- **(Mag Per Cond):** $u(x_1 + R, x_2) = e^{\frac{iRx_2}{2}} u(x_1, x_2)$ and $u(x_1, x_2 + R) = e^{\frac{-iRx_1}{2}} u(x_1, x_2)$ **will ensure periodicity of** $|u(x_1, x_2)|$, $|(\nabla - i\mathbf{A}_0)u(x_1, x_2)|$ and $\mathrm{Im}\Big(\overline{u}\,(\nabla - i\mathbf{A}_0)u\Big)$

Magnetic Periodic Condition (Mag Per Cond):

$$\mathbf{A}_0(x_1 + R, x_2) = \mathbf{A}_0(x_1, x_2) + \frac{R}{2}\nabla(x_2)$$

$$\mathbf{A}_0(x_1, x_2 + R) = \mathbf{A}_0(x_1, x_2) + \frac{R}{2}\nabla(-x_1)$$



$$u(x_1 + R, x_2) = e^{\frac{iRx_2}{2}} u(x_1, x_2)$$

$$u(x_1, x_2 + R) = e^{\frac{-iRx_1}{2}} u(x_1, x_2)$$

The Operator with Periodic Conditions:

- **The Hilbert space**: (with the inner product of $L^2(Q_R)$)

  $$L^2_{\text{per},\text{mag}}(Q_R) = \{u \in L^2_{\text{loc}}(\mathbb{R}^2) \ : \ \text{Mag Per Cond}\}$$

- **The form domain**: $\{u, (\nabla - i\mathbf{A}_0)u \in L^2_{\text{per},\text{mag}}\}$

- **The operator:** $P_R = -(\nabla - i\mathbf{A}_0)^2$

- **The spectrum:** $\{\mu_n = 2n - 1 \ : \ n \in \mathbb{N}\}$, when $R^2 \in 2\pi\mathbb{N}$

- **The advantage:** The space $E_R = \text{Ker}(P_R - \mu_1)$ is finite dimensional

The Abrikosov Energy:

- **The functional:** $\mathcal{E}_{\mathsf{Ab}}(v) = \int_{Q_R} \frac{1}{2}|v|^4 - |v|^2$

- **The space:** $v \in E_R = \{u \in L^2_{\mathsf{per,mag}} \ : \ P_R u = u\}$

- $\mathsf{E}_{\mathsf{gs,Ab}}(R) = \inf\{\mathcal{E}_{\mathsf{Ab}}(v) \ : \ v \in E_R\}$

- **Minimizers:** exist because $\dim E_R < \infty$

- **Behavior for R large:** $\lim_{R\to\infty} \frac{\mathsf{E}_{\mathsf{gs,Ab}}(R)}{|Q_R|} = E_{\mathsf{Ab}} \in (-\frac{1}{2}, 0)$

The Bulk Energy:

- **The functional:** (Given $b \in (0,1]$)

$$\mathcal{E}_{\mathsf{Blk}}(v) = \int_{Q_R} b|(\nabla - i\mathbf{A}_0)v|^2 - |v|^2 + \frac{1}{2}|v|^4$$

- **The space:** $v \in \Box$ (*Many choices !*)
  (i) (Dirichlet) $\Box = H_0^1(Q_R)$
  (ii) (Neumann) $\Box = H^1(Q_R)$
  (iii) (Mag Per Cond)

- $\mathsf{E}_{\mathsf{gs,Blk}}(b;R) = \inf\{\mathcal{E}_{\mathsf{Blk}}(v)\ :\ v \in \Box\}$

The Bulk Energy:

- **The functional:** (Given $b \in (0,1]$)
$\mathcal{E}_{\mathsf{Blk}}(v) = \int_{Q_R} b|(\nabla - i\mathbf{A}_0)v|^2 - |v|^2 + \frac{1}{2}|v|^4$

- $\mathrm{E}_{\mathsf{gs,Blk}}(b; R) = \inf\{\mathcal{E}_{\mathsf{Ab}}(v) \ : \ v \in \square\}$

- **Behavior for R large:** $\lim_{R\to\infty} \frac{\mathrm{E}_{\mathsf{gs,Blk}}(b;R)}{|Q_R|} = E_{\mathsf{Blk}}(b) \in (-\frac{1}{2}, 0]$

- $E_{\mathsf{Blk}}(\cdot)$ is **increasing**; $E_{\mathsf{Blk}}(b)\Big|_{b=1} = 0$.

From **Bulk** to **Abrikosov** Energy:

- (Aftalion-Serfaty and Fournais-Kachmar)

$$\frac{d}{db}\Big(\frac{E_{\mathsf{Blk}}(b)}{b-1}\Big)\Big|_{b=1_-} = \lim_{b\to 1_-} \frac{E_{\mathsf{Blk}}(b)}{(b-1)^2} = E_{\mathsf{Ab}}$$

- (Abrikosov) The key is to write $v = \sqrt{1-b}\,u$ and u an **eigenfunction** of the periodic operator $(-\nabla^2_{\mathbf{A}_0} u = u)$. We see that:

$$\begin{aligned}\mathcal{E}_{\mathsf{Blk}}(v) &= \int_{Q_R} b|(\nabla - i\mathbf{A}_0)v|^2 - |v|^2 + \frac{1}{2}|v|^4 \\ &= (b-1)^2 \int_{Q_R} \frac{1}{2}|u|^4 - |u|^2 = (b-1)^2 \mathcal{E}_{\mathsf{Ab}}(u)\end{aligned}$$

Density and Abrikosov Energy:

- Let v_{Blk} be a minimizer for $\mathcal{E}_{\mathsf{Blk}}(v)$
  $\mathcal{E}_{\mathsf{Blk}}(v_{\mathsf{Blk}}) \approx |Q_R|\, E_{\mathsf{Blk}}(b)$

- The equation $-b(\nabla - i\mathbf{A}_0)^2 v_{\mathsf{Blk}} = (1 - |v_{\mathsf{Blk}}|^2) v_{\mathsf{Blk}}$ yields

$$\frac{1}{2}\int_{Q_R} |v_{\mathsf{Blk}}|^4 = -\int_{Q_R} |(\nabla - i\mathbf{A}_0) v_{\mathsf{Blk}}|^2 - |v_{\mathsf{Blk}}|^2 + \frac{1}{2}|v_{\mathsf{Blk}}|^4$$
$$= -\mathcal{E}_{\mathsf{Blk}}(v_{\mathsf{Blk}}) \approx -|Q_R|\, E_{\mathsf{Blk}}(b)$$

- As $b \to 1_-$, the approximation $E_{\mathsf{Blk}}(b) \approx (b-1)^2 E_{\mathsf{Ab}}$ yields,

$$\frac{1}{2}\int_{Q_R} |v_{\mathsf{Blk}}|^4 \approx -(b-1)^2 |Q_R|\, E_{\mathsf{Ab}}$$

Density and Abrikosov Energy:

- Let u_{Ab} be a minimizer for $\mathcal{E}_{\mathsf{Ab}}(u)$
  $\mathcal{E}_{\mathsf{Ab}}(u_{\mathsf{Ab}}) \approx |Q_R|\, E_{\mathsf{Ab}}$

- The definition of $\mathcal{E}_{\mathsf{Ab}}(u) = \int_{Q_R} \frac{1}{2}|u|^2 - |u|^4$ yields

$$\int_{Q_R} |u_{\mathsf{Ab}}|^2 = -\mathcal{E}_{\mathsf{Ab}}(u_{\mathsf{Ab}}) + \frac{1}{2}\int_{Q_R} |u_{\mathsf{Ab}}|^4$$
$$\approx -|Q_R|\, E_{\mathsf{Ab}} + \frac{1}{2}\int_{Q_R} |u_{\mathsf{Ab}}|^4$$

Density and Abrikosov Energy:

- Let v_{Blk} and u_{Ab} be minimizers for $\mathcal{E}_{\mathsf{Blk}}(v)$ and $\mathcal{E}_{\mathsf{Ab}}(u)$

- $\frac{1}{2}\int_{Q_R}|v_{\mathsf{Blk}}|^4 \approx -(b-1)^2|Q_R|\,E_{\mathsf{Ab}}$

- $\int_{Q_R}|u_{\mathsf{Ab}}|^2 \approx -|Q_R|\,E_{\mathsf{Ab}} + \frac{1}{2}\int_{Q_R}|u_{\mathsf{Ab}}|^4$

- As $b \to 1_-$, it is reasonable to assume $v_{\mathsf{Blk}} \approx \sqrt{1-b}\,u_{\mathsf{Ab}}$

- **Conclusion:** $\dfrac{1}{|Q_R|}\displaystyle\int_{Q_R}|v_{\mathsf{Blk}}|^2 \approx -\,E_{\mathsf{Ab}}$

Connection with the Full Ginzburg-Landau Energy (Energy estimate)

Ginzburg-Landau Functional:

$$\mathcal{G}(\psi,\mathbf{A}) = \int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) dx + (\kappa H)^2 \int_\Omega |1 - \operatorname{curl}\mathbf{A}|^2\, dx\,.$$

$$\Omega \subset \mathbb{R}^2, \quad \kappa > 0, \quad H > 0\,.$$

The ground State Energy:

$$\mathsf{E}_{\mathsf{gs}}(\kappa,H) = \inf\{\mathcal{G}(\psi,\mathbf{A}) \ : \ (\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C})\times H^1(\Omega;\mathbb{R}^2)\}\,.$$

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Theorem: (Sandier-Serfaty) Let $b \in (0,1]$ and $H = b\kappa$. As $\kappa \to \infty$,

$$\mathrm{E}_{\mathrm{gs}}(\kappa,H) = \kappa^2|\Omega|E_{\mathrm{Blk}}(b) + o(\kappa^2)$$

Connection with the Full Ginzburg-Landau Energy (Energy estimate)

Theorem: (Sandier-Serfaty) Let $b \in (0,1]$ and $H = b\kappa$.

$$\mathsf{E}_{\mathsf{gs}}(\kappa, H) = \kappa^2 |\Omega| E_{\mathsf{Blk}}(b) + o(\kappa^2)$$

For $b \approx 1$ **?** **(**$E_{\mathsf{Blk}}(b) \approx E_{\mathsf{Blk}}(1) = 0$**)**

Theorem: (Fournais-Kachmar) Suppose that $H = \kappa - \mu(\kappa)$ and $\sqrt{\kappa} \ll \mu(\kappa) \ll \kappa$, i.e.

$$\mu(\kappa) > 0\,, \quad \lim_{\kappa \to \infty} \frac{\mu(\kappa)}{\sqrt{\kappa}} = \infty, \quad \lim_{\kappa \to \infty} \frac{\mu(\kappa)}{\kappa} = 0\,.$$

As $\kappa \to \infty$,

$$\mathsf{E}_{\mathsf{gs}}(\kappa, H) = [\kappa - H]^2 |\Omega| E_{\mathsf{Ab}} + o\Big([\kappa - H]^2\Big)$$

This is consistent with $b = \frac{H}{\kappa}$; $b \approx 1$; $E_{\mathsf{Blk}}(b) \approx (1-b)^2 E_{\mathsf{Ab}}$; $[\kappa - H]^2 = \kappa^2\,[1 - \frac{H}{\kappa}]^2$

Connection with the Full GL Energy
(Density of Superconductivity)

Ginzburg-Landau Functional:

$$\mathcal{G}(\psi,\mathbf{A}) = \int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) dx + (\kappa H)^2 \int_\Omega |1 - \mathrm{curl}\,\mathbf{A}|^2\, dx\,.$$

The ground State Energy:

$$\mathsf{E}_{\mathsf{gs}}(\kappa,H) = \inf\{\mathcal{G}(\psi,\mathbf{A}) \ : \ (\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2)\}\,.$$

Theorem: (Kac) Suppose that $H = \kappa - \mu(\kappa)$ and $\sqrt{\kappa} \ll \mu(\kappa) \ll \kappa$. Let $(\psi,\mathbf{A})$ be a **minimizer** of GL. As $\kappa \to \infty$

$$\int_\Omega |\psi|^2\, dx = -2\left[1 - \frac{H}{\kappa}\right] E_{\mathsf{Ab}} + o\left(\left[1 - \frac{H}{\kappa}\right]\right)$$

Superconducting Surfaces - Vanishing Magnetic Fields:

For a superconducting surface of revolution $\mathcal{M}$, the energy is (Contreras-Sternberg)

$$\mathcal{E}_{\mathcal{M}}(\psi) = \int_{\mathcal{M}} |(\nabla_{\mathcal{M}} - i\kappa H \mathbf{A}_e)\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4$$

$\mathbf{A}_e(x_1, x_2, x_3) = (-\frac{x_2}{2}, \frac{x_1}{2}, 0)$
$\text{curl}\,\mathbf{A}_e = \beta = (0, 0, 1)$
$B = \text{curl}_{\mathcal{M}}\,\mathbf{A}_e = (\beta \cdot \mathbf{N}) \times \text{non} - \text{vanishing function}$
$\mathbf{N} =$ unit normal vector of $\mathcal{M}$

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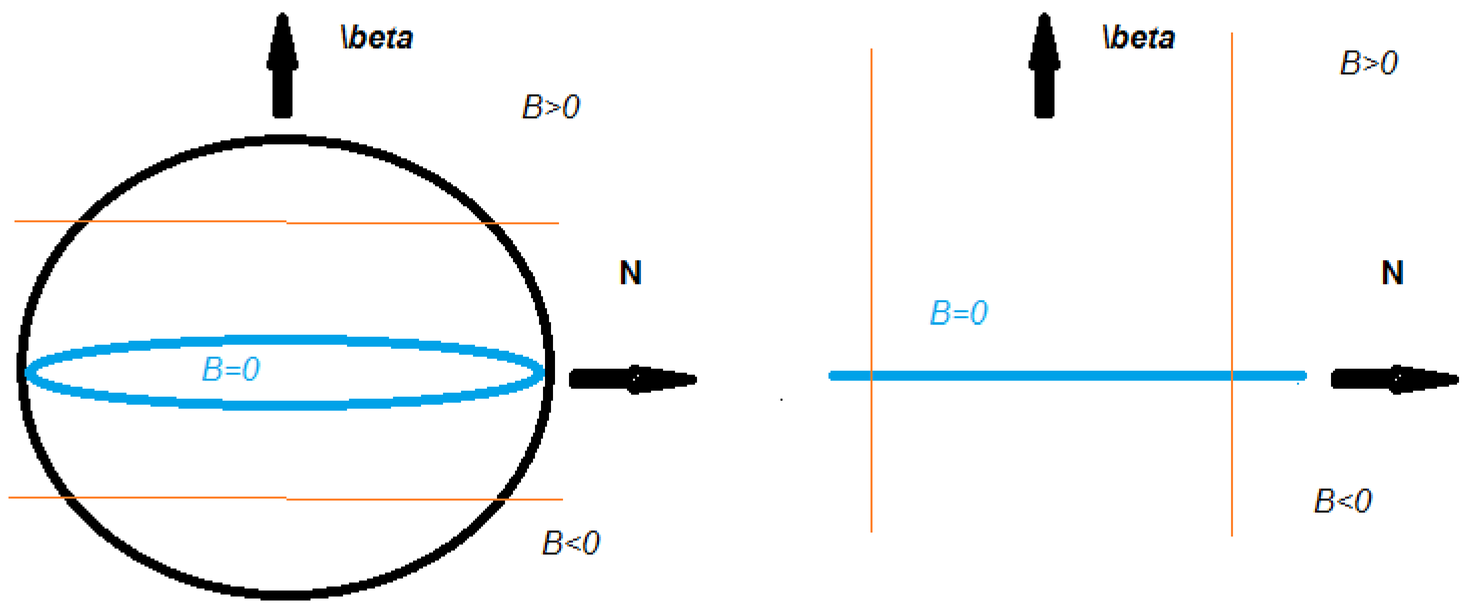

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A simple model is given by

$$\mathcal{E}_{L,R}(u) = \int_{(-R,R)\times\mathbb{R}} |(\nabla - i\mathbf{A}_{\mathrm{mod}})u|^2 - L^{-2/3}|u|^2 + \frac{L^{-2/3}}{2}|u|^4$$

$$\mathbf{A}_{\mathrm{mod}}(x) = \left(-\frac{x_2^2}{2}, 0\right), \qquad B = \operatorname{curl} \mathbf{A}_{\mathrm{mod}} = x_2$$

$$R \gg 1, \qquad L > 0.$$

From Constant to Vanishing Magnetic Fields:

The ground state energy for the simple model on a surface is

$$\mathfrak{e}(L;R) = \inf_u \mathcal{E}_{L,R}(u)\,,$$

The following limit exists (Helffer-Kachmar),

$$\mathfrak{e}(L) = \lim_{R\to\infty} \frac{\mathfrak{e}(L;R)}{2R}$$

As $L \to 0_+$, $\mathfrak{e}(L)$ is obtained from the bulk energy,

$$\mathfrak{e}(L) = L^{-4/3}\int_0^1 E_{\mathsf{Blk}}(b)\,db + o(L^{-4/3})$$

Central Role of Bulk Energy:

Abrikosov Energy

$\uparrow$ derivative of $\frac{E_{\mathsf{Blk}}(b)}{b-1}$ and $b \to 1_-$

$E_{\mathsf{Blk}}(b) \longrightarrow$ $integral\ of\ E_{\mathsf{Blk}}(b)$ superconducting surfaces

$\downarrow$ $b \to 0_+$

$Sandier - Serfaty\ (vortices)$