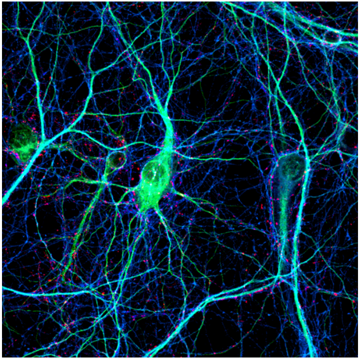
Second-order pseudo-stationary random fields and point processes on graphs and their edges

Jesper Møller
(in collaboration with Ethan Anderes and Jakob G. Rasmussen)

Aalborg University

Graph with edges = dendrite networks of neurons:



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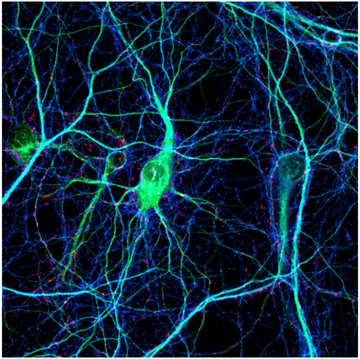

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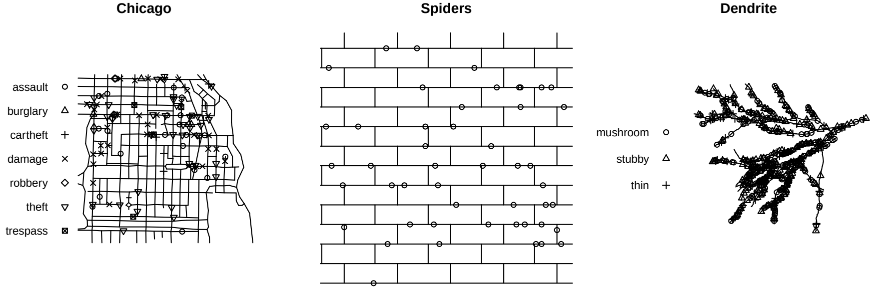
How do we model the random field = diameter along this graph with edges (i.e. all green lines!)?

Point patterns on graphs with edges (i.e. all lines!):

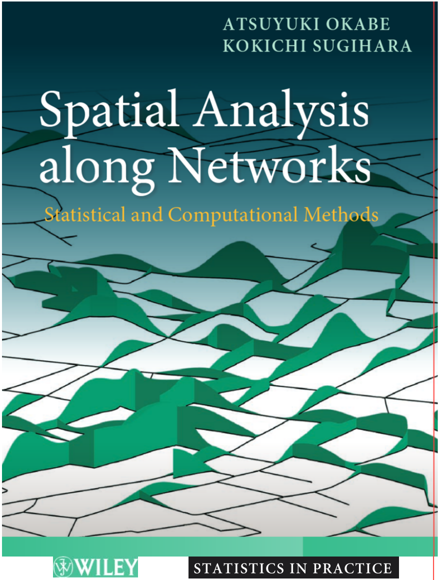


How do we determine
- clustering in street crimes?
- any evidence of interaction between positions of spider webs on mortar lines of a brick wall?
- the joint spatial distribution of spines (small protusions) of different types?

Snow's (1855) cholera map: Point pattern on a graph with edges = street network around the Broad Street pump:

Conclusion: cause of the victims' illness was contamination of the water from the Broad Street pump.

Textbook on ...


ATSUYUKI OKABE
KOKICHI SUGIHARA
Spatial Analysis along Networks
Statistical and Computational Methods
WILEY
STATISTICS IN PRACTICE

Some other research:

Cressie, Frey, Harch & Smith (2006). Spatial prediction on a river network. *Journal of Agricultural, Biological, and Environmental Statistics*.

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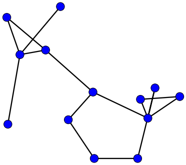
Ang, Baddeley & Nair (2012). Geometrically corrected second order analysis of events on a linear network, with applications to ecology and criminology. *Scandinavian Journal of Statistics*.

Baddeley, Jammalamadaka & Nair (2014). Multitype point process analysis of spines on the dendrite network of a neuron. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*.

Definition 1 (more general than seen elsewhere!):



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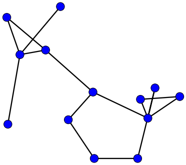

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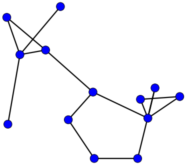

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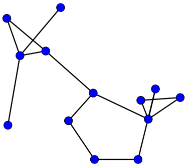

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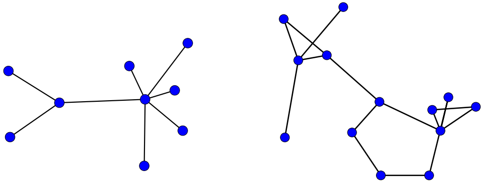
(c) $\varphi_i : e_i \mapsto (a_i, b_i)$ is a bijection (**edge-coordinate**).
E.g. φ_i^{-1} = natural parametrization of e_i.

L = index set for random fields/space for point processes on $\mathcal{G}$:



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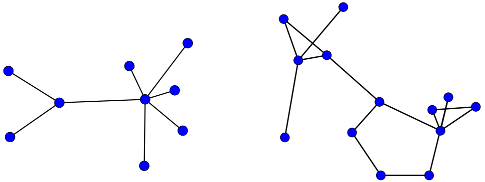

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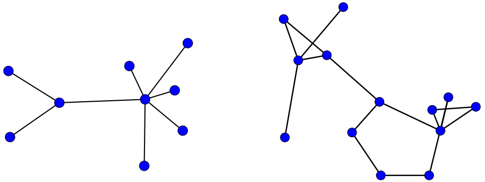

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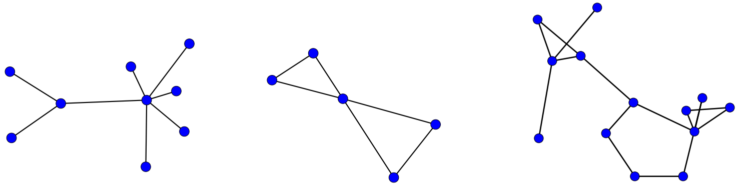
Geodesic distance: $d_{\mathcal{G}}(u, v) =$ **infimum of length of paths in** $\mathcal{G}$ **between** $u, v \in L$ (where "length" is induced by edge-coordinates and usual length on the intervals (a_i, b_i)).

(Existing literature consider only the special case of a) **linear network**: edges = straight line segments, only meeting at vertices, and $\varphi_i \sim$ natural parametrization, so
$d_{\mathcal{G}}(u, v) =$ length of shortest set-connected path between u and v.



(Left and middle panels: linear networks. Right panel: *not* a linear network.)

Open problems and motivations:

- How do we construct **covariance functions** of the form

$$c(u, v) = c_0(d_{\mathcal{G}}(u, v))$$

for $u, v \in L$? Say then that c is **pseudo-stationary**.

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- How do we construct **point processes on** L with pair correlation function of the form

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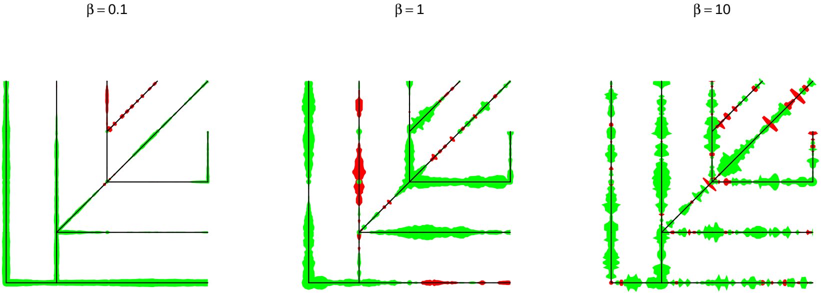
  So far only the Poisson process is known to be pseudo-stationary.

PART 1: PSEUDO-STATIONARY COVARIANCE FUNCTIONS AND RANDOM FIELDS



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• A graph with Euclidean edges $\mathcal{G}$ is said to **support the PDEFs** if for any $\beta > 0$,

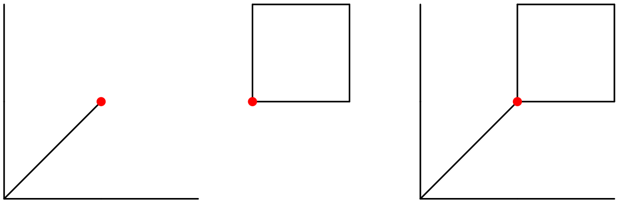
$$c(u, v) = \exp(-\beta d_{\mathcal{G}}(u, v))$$

is positive semi-definite for $u, v \in L$.

Definition 3:

Suppose $\mathcal{G}_1 = (\{\mathcal{V}_1, \{e_i : i \in I_1\}, \{\varphi_i : i \in I_1\})$ and $\mathcal{G}_2 = (\{\mathcal{V}_2, \{e_i : i \in I_2\}, \{\varphi_i : i \in I_2\})$ have only one vertex v_0 in common, but no common edges and disjoint index sets I_1 and I_2.

The **1-sum** of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the graph with Euclidean edges given by $\mathcal{G} = (\mathcal{V}_1 \cup \mathcal{V}_2, \{e_i : i \in I_1 \cup I_2\}, \{\varphi_i : i \in I_1 \cup I_2\})$.



Graphs with Euclidean edges supporting the exponential covariance function:

Theorem 1. If $\mathcal{G}_1, \mathcal{G}_2, \ldots$ support the PDEFs, then the 1-sum of $\mathcal{G}_1, \mathcal{G}_2, \ldots$ supports the PDEFs. In fact $\sigma^2 \exp(-\beta d_{\mathcal{G}}(u, v))$ is (strictly) positive definite for all $\beta, \sigma^2 > 0$.

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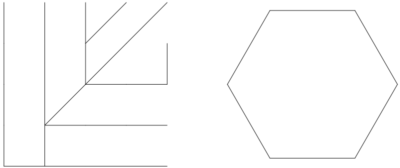
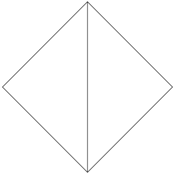
Theorem 2. Cycles and trees support the exponential covariance function, and so do countable 1-sums of these.



Forbidden subgraph:

Theorem 3. Suppose G is a graph with Euclidean edges that has three paths which have common endpoints but are otherwise pairwise disjoint.



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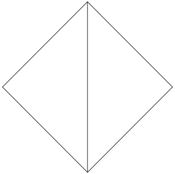

Then there exists a $\beta > 0$ s.t.

$$c(u, v) = \exp(-\beta d_{\mathcal{G}}(u, v)), \quad u, v \in L,$$

is **not** positive semi-definite.

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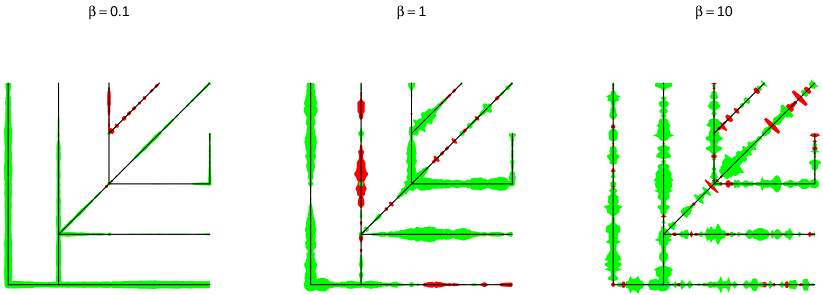
- On a finite collection of n points $\subset L$: "just" sim. from N_n.
- On a tree $\mathcal{G}$: 1) Simulate multivariate normal distribution on $\mathcal{V}$ (can be done sequentially).
  2) Exploit Markov property: Simulate conditional independent Ornstein-Uhlenbeck processes on edges given the values on $\mathcal{V}$.



Completely monotonic covariance functions:

$c_0 : [0, \infty) \mapsto [0, \infty)$ is **completely monotonic** if it is continuous and $(-1)^k c_0^{(k)}(t) \geq 0$ for all $t \in (0, \infty)$ and $k = 1, 2, \ldots$.

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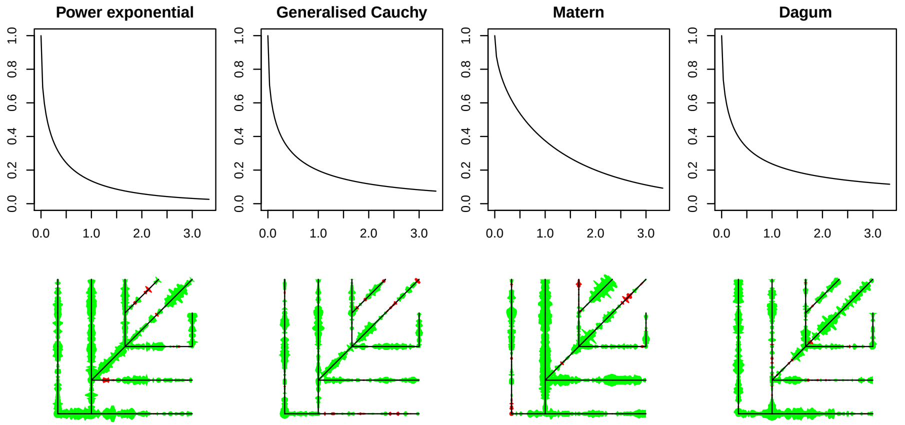
for some $\sigma^2 > 0$ and some non-constant r.v. $Y \geq 0$.
- Distribution of Y = inverse Laplace transform of $\mathcal{L}(t) = c_0(t)/\sigma^2$. If available on closed form, then simulation boils down to simulate
  - a realization $Y = \beta$
  - a GRF with $c(u,v) = \sigma^2 \exp(-\beta d_{\mathcal{G}}(u,v))$.

Simulations using completely monotonic covariance fcts:



Examples of completely monotonic covariance functions:

Theorem 5. Suppose $\mathcal{G}$ supports the PDEFs. Then for $\sigma^2, \beta > 0$, we have parametric families of pos. def. cov. fcts. $c(u,v) = c_0(d_{\mathcal{G}}(u,v))$:

- **Power exponential covariance function:**

$$c_0(s) = \sigma^2 \exp\left(-\beta s^\alpha\right), \quad \alpha \in (0,1].$$

- **Generalized Cauchy covariance function:**

$$c_0(s) = \sigma^2 \left(\beta s^\alpha + 1\right)^{-\xi/\alpha}, \quad \alpha \in (0,1],\ \xi > 0.$$

- **The Matérn covariance function:**

$$c_0(s) = \sigma^2 \frac{(\beta s)^\alpha K_\alpha(\beta s)}{\Gamma(\alpha) 2^{\alpha-1}}, \quad \alpha \in (0, 1/2].$$

- **The Dagum covariance function:**

$$c_0(s) = \sigma^2 \left[1 - \left(\frac{\beta s^\alpha}{1 + \beta s^\alpha}\right)^{\xi/\alpha}\right], \quad \alpha, \xi \in (0,1].$$

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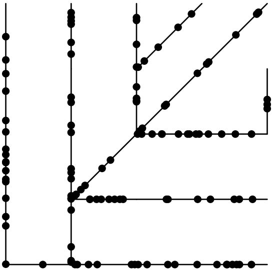
Theorem 6. For any of the functions $c(u, v)$ given in Theorem 5 but with $\alpha > 0$ outside the parameter range given in Theorem 5,

- there exists a graph with Euclidean edges $\mathcal{G}$ which supports the PDEFs (and is not necessarily a cycle),
- but $c(u, v)$ is **not** a covariance function.

PART 2: PSEUDO-STATIONARY POINT PROCESSES



Definitions for point processes on $\mathcal{G}$:

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- **Pair correlation function**: $g(u, v) = \rho^{(2)}(u, v)/[\rho(u)\rho(v)]$.

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• If $\rho(u)$ is locally integrable, then for each $u \in L$ there exists a point process $X_u^!$ on $\mathcal{G}$ which follows the **reduced Palm distribution at** u, i.e.

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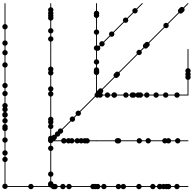
If $\rho(u) \equiv \rho$ and $g(u,v) = g_0(d_{\mathcal{G}}(u,v))$, then for any $u \in L$,

$$\begin{aligned}\rho K(r) &= \mathrm{E}\#\{v \in X_u^! : d_{\mathcal{G}}(u,v) \leq r\} \\ &= \mathrm{E}[\#\{(X \setminus \{u\}) \cap b_{d_{\mathcal{G}}}(u,r)\} \mid u \in X].\end{aligned}$$

Poisson processes:

• X is a **Poisson process** on $\mathcal{G}$ with (locally integrable) intensity function $\rho : L \mapsto [0, \infty)$, if for any $B \subseteq L$ with $\mu(B) := \int_B \rho(u)\, \mathrm{d}\lambda_{\mathcal{G}}(u) < \infty$,

- $\#(X \cap B) \sim \mathrm{Poisson}(\mu(B))$,
- cond. on $\#(X \cap B)$, the points in $X \cap B$ are iid with density $\propto \rho$.



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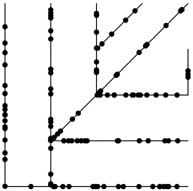

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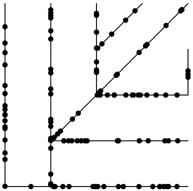

• Then $\rho^{(n)}(u_1, \ldots, u_n) = \rho(u_1) \cdots \rho(u_n)$, so $g(u, v) = 1$, i.e. X is pseudo-stationary and $K(r) = r$. Moreover, $X_u^! \sim X$ whenever $\rho(u) > 0$.

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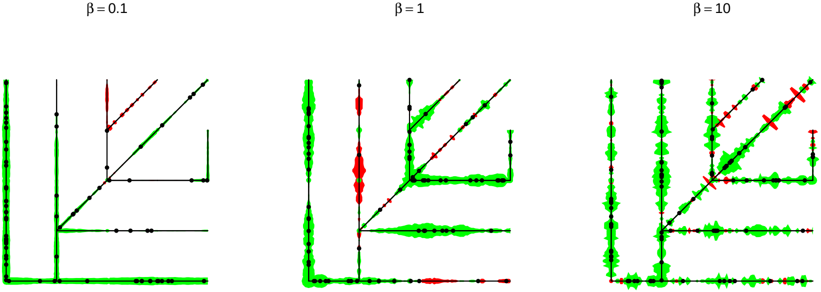
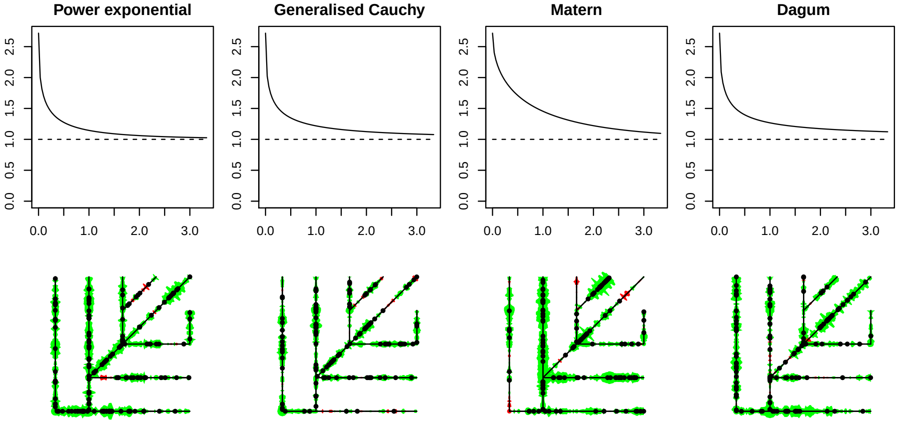
• $X_u^!$ is a LGCP with underlying GRF having mean function $m_u(v) = m(v) + c(u,v)$ and covariance function c.

Simulations of LGCPs using exponential covariance fcts:

Given a realisation of the GRF Z on $\mathcal{G}$, we simulate a Poisson process with intensity function $\exp(Z)$ to obtain a simulation of the LGCP X on $\mathcal{G}$.



Simulations of LGCPs using other covariance fcts:



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- For other graphs even something simple like the exponential function is not (necessarily) a covariance function.
- The covariance functions we have established are all completely monotonic, so they cannot e.g. be negative.

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THANK YOU!