Faculty of Science

Cointegrated Oscillating Systems

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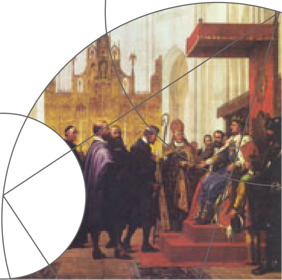
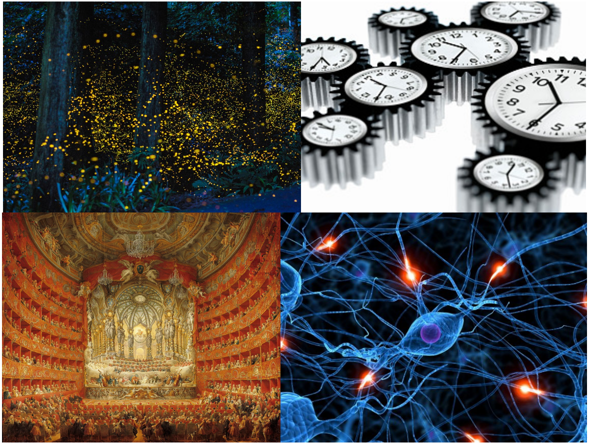


Agenda

1. Motivation
2. Cointegration
3. Oscillators
4. Simulation
5. Outlook

Motivation

Motivation: Synchronization in Nature



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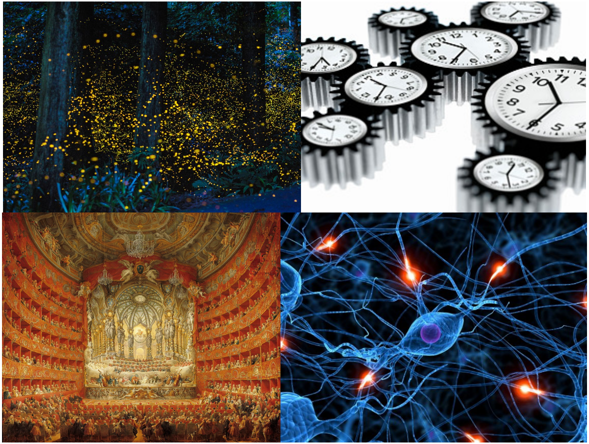

An ubiquitous presence...

Cointegration

Cointegration: Unit Roots

Consider a p-dim process

$$x_t = A_1 x_{t-1} + \cdots + A_k x_{t-k} + \varepsilon_t,$$

where $A_i \in \mathbb{R}^{p \times p}$ and $\varepsilon_t \in \mathbb{R}^p$.

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If $C(z) = 0$ for $|z| = 1$, the process contains a *unit root*, and it is *nonstationary*.

Cointegration: Unit Roots

1D example: simple random walk

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$$C(z) = 1 - z, z \in \mathbb{C}$$

has a root $|z| = 1$, x_t has a unit root of multiplicity 1, it is thus *integrated of order 1*: $I(1)$ and we say that x_t has a *stochastic trend*.

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A process with a unit root of multiplicity d is $I(d)$.

For x_t an $I(d)$ process, Δx_t is $I(d-1)$ thus $\Delta^d x_t$ is $I(0)$.

Cointegration: Vector Error Correction

Assume $x_t \in \mathbb{R}^p$ is $I(1)$ and rewrite

$$x_t = A_1 x_{t-1} + \cdots + A_k x_{t-k} + \varepsilon_t,$$

as

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \cdots + \Gamma_{k-1} \Delta x_{t-k+1} + \varepsilon_t,$$

where

$$\begin{aligned} \Pi &= -(I_p - A_1 - \cdots - A_k) \\ \Gamma_j &= -(A_{j+1} + \cdots + A_k), \quad j = 1, \dots k-1. \end{aligned}$$

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If x_t is $I(1)$ then Δx_t is $I(0)$ and thus Πx_{t-1} must be $I(0)$!

Cointegration: Vector Error Correction

3 possibilities for $\text{rank}(\Pi) = r$:

- Π has full rank p.
- Π has reduced rank $0 < r < p$.
- Π has rank 0.

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- Π has full rank p.
- Π has reduced rank $0 < r < p$.
- Π has rank 0.

$r = p \Rightarrow$ that x_t *must* be $I(0)$.

$r = 0 \Rightarrow$ no stationary relations of x_t.

$0 < r < p \Rightarrow r$ stationary combinations of x_t variables.
We then say that x_t is a *cointegrated* process.

Cointegration: Parameters

If Π has rank $0 < r < p$, then

$$\Pi = \alpha\beta',$$

where $\alpha, \beta \in \mathbb{R}^{p \times r}$ and rank $r < p$.

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Hence the r linearly independent columns of β correspond to r stationary linear combinations of x_t.

Note also that α and β are not uniquely identified!
This implies only linear restrictions as hypotheses.

Cointegration: Dissection of Trends

Consider the simple example

$$\begin{aligned} x_t &= (x_{1t}, x_{2t})' \\ \Delta x_t &= \alpha\beta' x_{t-1} + \varepsilon_t \\ \beta &= (1, -1)', \end{aligned}$$

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Here $\beta_\perp = (1, 1)'$ is the orthogonal complement to β, such that $\beta'\beta_\perp = 0$.

Same concept for $\alpha_\perp$.

$\alpha_\perp' \sum_{i=1}^{t} \varepsilon_i$ are stochastic trends,
$\beta' x_t$ are stationary trends.

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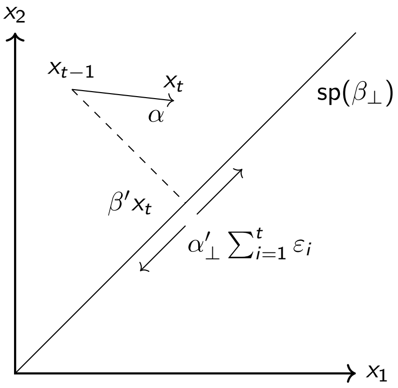

Cointegration: $\Delta x_t \to dx_t$

Consider the diffusions

$$dx_t = \Pi x_t dt + dW_t$$
$$d\tilde{x}_t = \widetilde{\Pi}\tilde{x}_t dt + d\widetilde{W}_t.$$

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An example from [Kessler & Rahbek, 2001] show that with

$$\Pi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \widetilde{\Pi} = \begin{pmatrix} 0 & -2\pi/\delta \\ 2\pi/\delta & 0 \end{pmatrix},$$

then

$$\exp(\Pi\delta) = \exp(\widetilde{\Pi}\delta) = I_2, \quad \delta \in \mathbb{R}.$$

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And also that the diffusion terms for x_t and $\tilde{x}_t$ are indistinguishable.

Cointegration: $\Delta x_t \to dx_t$

Hence, the sampling frequency of the data, can affect whether we can conclude on a model with $\text{rank}(\Pi) = 0$ or $\text{rank}(\Pi) = 2$.

Cointegration: $\Delta x_t \rightarrow dx_t$

Hence, the sampling frequency of the data, can affect whether we can conclude on a model with rank(Π) $= 0$ or rank(Π) $= 2$.

However for reduced rank matrices $\Pi = \alpha\beta'$ and $\widetilde{\Pi} = ab'$, they conclude that

$$\begin{aligned} \text{sp}(\alpha) &= \text{sp}(a) \\ \text{sp}(\beta) &= \text{sp}(b). \end{aligned}$$

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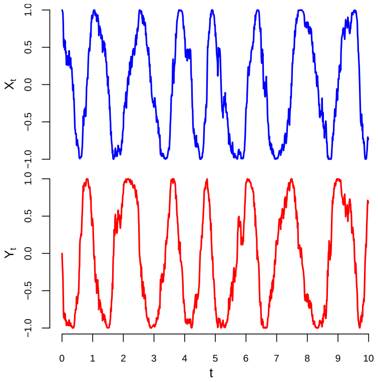
Thus we use *Johansens* method for determining $\text{rank}(\Pi)$, then β and simultaneously provides MLE for the remaining parameters.

Oscillators

Oscillators: A Definition

Assume a bivariate process $z_t = (x_t, y_t)'$, such that we observe something like this



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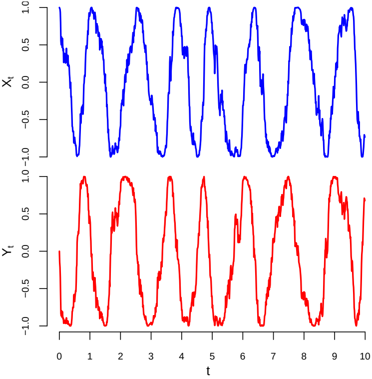

We define the *phase process* $\phi_t \in \mathbb{R}$ through the SDE

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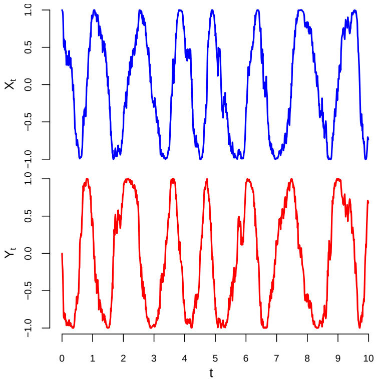

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We can now define z_t as

$$x_t = \gamma_t \cos(\phi_t)$$
$$y_t = \gamma_t \sin(\phi_t),$$

for some none-negative amplitude process γ_t.

Oscillators: Multivariate Phase Process

We generalize to a system of p oscillators, with phase/amplitude-processes $\phi_t, \gamma_t \in \mathbb{R}^p$:

$$d\phi_t = \big(f(\phi_t) + \mu_t\big)dt + \Sigma dW_t, \tag{1}$$

with $f(\phi_t) : \mathbb{R}^p \to \mathbb{R}^p$, $\mu_t \in \mathbb{R}^p$, $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p)$ and dW_t a p-dimensional Wiener process.

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Compare with the Kuramoto model, a classical model of coupled phases:

$$d\phi_{it} = \Big(\frac{\alpha_i}{p}\sum_{j=1}^{p}\sin(\phi_{jt} - \phi_{it}) + \mu_i\Big)dt + \sigma_i dW_{it}, i = 1, \ldots, p.$$

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Oscillating Systems: Cointegrated Phases

So we make the following assumption

$$f(\phi_t) = \Pi\phi_t = \alpha\beta'\phi_t$$

where $\text{rank}(\Pi) = r < p$, such that $\alpha, \beta \in \mathbb{R}^{p \times r}$ have full column rank.

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where $\text{rank}(\Pi) = r < p$, such that $\alpha, \beta \in \mathbb{R}^{p\times r}$ have full column rank.

With this assumption, ϕ_t is a *cointegrated process* and we derive the data generating process

$$dz_{it} = \begin{pmatrix} \frac{-\sigma_i^2}{2} & -(g(z_t)_i + \mu_i) \\ g(z_t)_i + \mu_i & \frac{-\sigma_i^2}{2} \end{pmatrix} z_{it}dt + \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} z_{it}dW_{it} + z_{it}\frac{d\gamma_{it}}{\gamma_{it}},$$

where dW_{it} and $d\gamma_{it}$ are uni-variate processes, and

$$g(z_t) = f(\phi_t) = \alpha\beta'\phi_t \quad \text{is } p \times 1$$

such that $g(z_t)_i$ denotes the i'th component of $g(z_t)$.

Simulation

Simulation: Unwrapped phases

We solve for z_t numerically, then obtain ϕ_t as

$$\phi_{it} = \text{atan2}(y_{it}, x_{it}) + 2\pi k_{it},$$

with k_{it} the number of rotations at time t for z_{it}, and $\text{atan2}(y_{it}, x_{it}) \in [0, 2\pi)$.

This way we get the *unwrapped* phases $\phi_t \in \mathbb{R}^p$.

Simulation: Models

Four different systems

$$\Pi_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Pi_1 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Pi_2 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Pi_3 = \begin{pmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{pmatrix}$$

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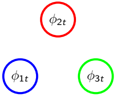
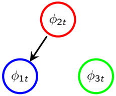
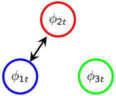
Π_0 :



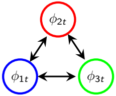
Π_1 :



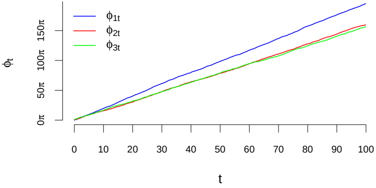
Π_2 :



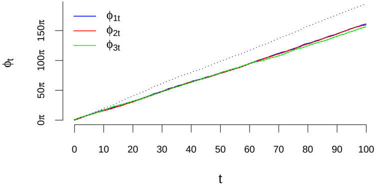
Π_3 :



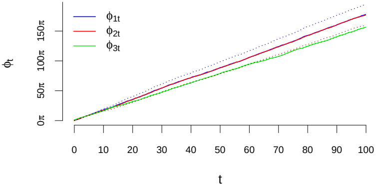
Simulation: Phases



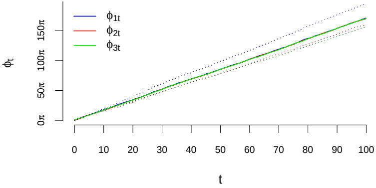
(a) Π_0 model.



(b) Π_1 model.



(c) Π_2 model.



(d) Π_3 model.

Simulation: Rank tests for rank(Π)

Johansen rank tests.

Model	H_r	Test values	p-value
Π_0	$r = 0$	14.94	0.751
	$r \leq 1$	6.73	0.519
	$r \leq 2$	0.17	0.635
Π_1	$r = 0$	52.50	0.000
	$r \leq 1$	5.61	0.489
	$r \leq 2$	0.78	0.306
Π_2	$r = 0$	64.78	0.000
	$r \leq 1$	6.57	0.305
	$r \leq 2$	0.00	0.983
Π_3	$r = 0$	77.39	0.000
	$r \leq 1$	33.24	0.000
	$r \leq 2$	0.01	0.899

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Johansen rank tests.

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	$r \leq 1$	6.73	0.519
	$r \leq 2$	0.17	0.635
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	$r \leq 1$	5.61	0.489
	$r \leq 2$	0.78	0.306
Π_2	$r = 0$	64.78	0.000
	$r \leq 1$	6.57	0.305
	$r \leq 2$	0.00	0.983
Π_3	$r = 0$	77.39	0.000
	$r \leq 1$	33.24	0.000
	$r \leq 2$	0.01	0.899

Simulation: Π_1 model

Fitted model Π_1 with unrestricted α, β:

Parameter	True value	Unrestricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.471	0.072	< 0.001
α_2	0	0.074	0.075	0.329
α_3	0	-0.121	0.077	0.117
β_1	1	1		
β_2	-1	-1.028		
β_3	0	0.031		
μ_1	6	6.321	0.214	< 0.001
μ_2	5	4.810	0.224	< 0.001
μ_3	5	5.209	0.230	< 0.001

Simulation: Π_1 model

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$$H_{\alpha,\beta}: \quad \alpha = A\psi, \text{ with } A = (1,0,0)'$$
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Fitted model Π_1 under $H_{\alpha,\beta}$ (p-value is 0.365):

Parameter	True value	Restricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.469	0.072	< 0.001
α_2	0	0		
α_3	0	0		
β_1	1	1		
β_2	-1	-1		
β_3	0	0		
μ_1	6	6.066	0.180	< 0.001
μ_2	5	5.006	0.188	< 0.001
μ_3	5	4.886	0.193	< 0.001

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μ_1	6	6.066	0.180	< 0.001
μ_2	5	5.006	0.188	< 0.001
μ_3	5	4.886	0.193	< 0.001

Conclusion: We recover the correct uni-directional coupling structure.

Simulation: Π_2 model

Fitted model Π_2 with unrestricted α, β:

Parameter	True value	Unrestricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.437	0.103	< 0.001
α_2	0.5	0.593	0.105	< 0.001
α_3	0	-0.190	0.110	0.083
β_1	1	1		
β_2	-1	-0.989		
β_3	0	-0.012		
μ_1	6	6.160	0.164	< 0.001
μ_2	5	4.777	0.167	< 0.001
μ_3	5	5.134	0.176	< 0.001

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Fitted model Π_2 under $H_{\alpha,\beta}$ (p-value is 0.187):

Parameter	True value	Unrestricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.497	0.101	< 0.001
α_2	0.5	0.497	0.103	< 0.001
α_3	0	0		
β_1	1	1		
β_2	-1	-1		
β_3	0	0		
μ_1	6	6.129	0.144	< 0.001
μ_2	5	5.013	0.148	< 0.001
μ_3	5	4.886	0.155	< 0.001

Simulation: Π_2 model

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Simulation: Π_3 model

Fitted model Π_3 with unrestricted α, β:

Parameter	True value	Unrestricted β Estimate	Std. Error	p value
α_{11}	-0.50	-0.248	0.074	0.001
α_{21}	0.25	0.374	0.064	< 0.001
α_{31}	0.25	0.184	0.076	0.015
α_{12}	0.25	0.226	0.065	0.001
α_{22}	-0.50	-0.033	0.078	0.673
α_{32}	0.25	-0.301	0.067	< 0.001
β_{11}	1	1		
β_{21}	0	-1.409		
β_{31}	-1	0.410		
β_{12}	0	-0.865		
β_{22}	1	0.344		
β_{32}	-1	1.209		
μ_1	6	6.196	0.194	< 0.001
μ_2	5	4.712	0.199	< 0.001
μ_2	5	5.152	0.204	< 0.001

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Simulation: Π_3 model

Look at the estimated $\hat{\Pi}$:

$$\hat{\Pi} = \hat{\alpha}\hat{\beta}' = \begin{pmatrix} -0.444 & 0.272 & 0.171 \\ 0.346 & -0.539 & 0.193 \\ 0.076 & 0.363 & -0.439 \end{pmatrix}$$

vs

$$\Pi_3 = \begin{pmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{pmatrix}$$

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- $\sum_{j=1}^{3} \hat{\Pi}_{ij} \approx 0$.
- Identification of α, β can be problematic...

Simulation: Conclusions

Cointegration analysis findings:

Model	Description	Conclusion
Π_0	Independent	Independent oscillators.
Π_1	Uni-directional coupling	Uni-directional coupling.
Π_2	Bi-directional coupling	Bi-directional coupling, equal coupling strength.
Π_3	Fully coupled	Unclear*.

*: We cannot identify the true parameters for the model fit, but the data does admit a restriction to the true proportions of the parameter matrices.

Outlook

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- Derive a framework with non-linear deterministic trends for the model.

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From a practical perspective, it is easier to approximate $\sin(x)$ for $x \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, since then

$$\sin(x) \approx \begin{cases} x & \text{for } x \leq \frac{\pi}{2} \\ -x + \pi & \text{for } x > \frac{\pi}{2} \end{cases}.$$

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Wrapping $\beta' \phi_t$ onto $[-\frac{\pi}{2}, \frac{3\pi}{2})$ yields two cointegration regimes:

$$\alpha \sin(\beta' \phi_t) \approx \begin{cases} \alpha \beta' \phi_t & \text{for } \beta' \phi_t \leq \frac{\pi}{2} \\ -\alpha \beta' \phi_t + \alpha \pi & \text{for } \beta' \phi_t > \frac{\pi}{2} \end{cases} .$$

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This is work in progress to derive the necessary assumptions and technicalities!

?

Thank you!

Jacob Østergaard

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