Eigenvalues of Semi-classical Neumann Magnetic Laplacian and comparison with Robin Laplacian

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Plan

The magnetic Laplacian

The geometry:

- $\mathbf{B} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ a regular magnetic field.
- $\mathbf{A} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ a magnetic potential satisfying $\text{curl}\,\mathbf{A} = \mathbf{B}$.
- Ω a simply connected subset of $\mathbb{R}^3$.

Semiclassical Magnetic Laplacian:

$$H(\mathbf{A}, \Omega)[h] := (-ih\nabla - \mathbf{A})^2 \quad \text{on } \Omega \quad \text{with} \quad h > 0.$$

- Magnetic Neumann boundary conditions:

$$\mathbf{n} \cdot (-ih\nabla - \mathbf{A})u = 0 \text{ on } \partial\Omega.$$

- Associated quadratic form: $u \mapsto \int_\Omega |(-ih\nabla - \mathbf{A})u|^2 \mathrm{d}x$.
- $H(\mathbf{A}, \Omega)[h]$ is positive self-adjoint.
- If Ω is Lipschitz and bounded, the form domain is $H^1(\Omega)$, and $H(\mathbf{A}, \Omega)[h]$ has compact resolvent.

Goals

Gauge invariance:

- The spectrum depends only on $\mathbf{B} = \operatorname{curl} \mathbf{A}$.
- $\lambda_h(\mathbf{B}, \Omega)$ the first eigenvalue.

Behavior of $\lambda_h(\mathbf{B}, \Omega)$ when h goes to 0:

- The influence of the geometry of Ω and the magnetic field $\mathbf{B}$.
- The localization of the eigenfunctions associated with $\lambda_h(\mathbf{B}, \Omega)$ when h goes to 0.

Link with the spectrum for large magnetic fields:

$$H(\mathbf{A}, \Omega)[h] = h^2 H\left(\frac{\mathbf{A}}{h}, \Omega\right) \quad \text{avec} \quad H\left(\breve{\mathbf{A}}, \Omega\right) := (-i\nabla - \breve{\mathbf{A}})^2$$

Application to surface superconductivity.

S. Fournais and B. Helffer.
Spectral methods in surface superconductivity.
Progress in Nonlinear Differential Equations and their Applications (2010)

Natural scaling

Standard elementary example:

- $\Omega = \mathbb{R}^3$ et $\mathbf{B} = (0, b, 0)$.
- Let $\mathbf{A}(x_1, x_2, x_3) := b(\frac{x_3}{2}, 0, -\frac{x_1}{2})$ satisfying curl $\mathbf{A} = (0, b, 0)$.

$$H(\mathbf{A}, \mathbb{R}^3)[h] = \left(-ih\partial_{x_1} - b\frac{x_3}{2}\right)^2 - h^2\partial_{x_2}^2 + \left(-ih\partial_{x_3} + b\frac{x_1}{2}\right)^2 \text{ sur } \mathbb{R}^3 .$$

- "Semiclassical" scaling :

$$X = \frac{1}{\sqrt{h}} x$$

We find

$$H(\mathbf{A}, \mathbb{R}^3)[h] \simeq h H(\mathbf{A}, \mathbb{R}^3) .$$

- Valid for any conical domain.

Results in dimension 2

B a scalar non-vanishing magnetic field. Let

$$b = \inf_{x\in\Omega} |B(x)| \quad \text{and} \quad b' = \inf_{x\in\partial\Omega} |B(x)| \quad \text{with} \quad b \neq 0\,.$$

Asymptotic expansion in dimension 2 [Lu-Pan 99], [Bonnaillie 2005]

$$\text{Regular case :}\ \lambda_h(B,\Omega) \underset{h\to 0}{\sim} h \min\left\{b,\ b'\Theta_0\ \right\}.$$

$$\text{Polygonal case :}\ \lambda_h(B,\Omega) \underset{h\to 0}{\sim} h \min\left\{b,\ b'\Theta_0, \min_{\mathbf{v}} |B(\mathbf{v})|\,\mu(\alpha(\mathbf{v}))\right\}$$

with $\mathbf{v} \in \overline{\Omega}$ the vertices of opening $\alpha(\mathbf{v})$

- $\Theta_0 \approx 0.5901$ bottom of the spectrum of a model problem on a half-plane $\mathbb{R}^2_+$ (de Gennes 62).
- $\mu(\alpha) \leq \Theta_0$ bottom of the spectrum of a model problem on the infinite sector $\mathcal{S}_\alpha$ of opening α.

Magnetic fields in 3d regular domains

Let $\sigma(\theta)$ be the ground energy of the model operator $H(\mathbf{A}_\theta, \mathbb{R}^3_+)$ with

- $\mathbb{R}^3_+ = \{(x_1, x_2, x_3),\ x_1 > 0\}$ the model half-space.
- $\operatorname{curl} \mathbf{A}_\theta = \mathbf{B}_\theta := (\sin\theta, \cos\theta, 0)$ makes an angle θ with the boundary.

Theorem [Lu–Pan 2000, Helffer-Morame, 2004]

Let Ω be a regular domain. For $x \in \partial\Omega$, let $\theta(x)$ the angle between $\partial\Omega$ and $\mathbf{B}$ at x.

$$\lambda_h(\mathbf{B}, \Omega) \underset{h\to 0}{\sim} h \min\left\{ \inf_{x\in\Omega} |\mathbf{B}(x)|, \inf_{x\in\partial\Omega} |\mathbf{B}(x)|\, \sigma(\theta(x)) \right\}$$

- $\theta \mapsto \sigma(\theta)$ is increasing on $[0, \frac{\pi}{2}]$ with $\sigma(0) = \Theta_0$ and $\sigma(\frac{\pi}{2}) = 1$.
- Corollary: if $\mathbf{B}$ is constant, the minimum is Θ_0 and corresponds to the point of Ω at which the magnetic field is tangent.

Theorem: Cuboid [Pan 02]

Let $\mathcal{C}$ be a cuboid. Then there exists an octant Π such that:

$$\lambda_h(\mathbf{B}, \mathcal{C}) \underset{h\to 0}{\sim} hE(\mathbf{B}, \Pi) \quad \text{with} \quad E(\mathbf{B}, \Pi) < \Theta_0 \,.$$

Objective

Objectives of this talk:

- Find the first term of the asymptotics for general domains and understand the hierarchy of model problems:

$$\lambda_h(\mathbf{B}, \Omega) = h\mathscr{E}(\Omega, \mathbf{B}) + O(h^\kappa)$$

- Find more terms in the asymptotics, study the higher eigenvalues λ_h^k and the structure of the spectrum:

$$\lambda_h^k(\mathbf{B}, \Omega) = h\mathscr{E}(\Omega, \mathbf{B}) + \sum_j \gamma_{j,k} h^{\kappa_j}.$$

  Give sufficient geometrical conditions to see the influence of k?

- Compare with the analysis of Robin Laplacians:

$$\begin{cases} -\Delta u = \lambda u \text{ on } \Omega \\ \partial_n u - \alpha u = 0 \text{ on } \partial\Omega \end{cases} \quad \text{with } \alpha \to +\infty.$$

Plan

Corner domains and tangent cones in dimension 3

With each point $x \in \overline{\Omega}$ is associated its tangent cone Π_x whose section by $\mathbb{S}^2$ is a curvilinear polygon.

Situation of $x \in \overline{\Omega}$	Model geometry Π_x
Interior point	Space $\mathbb{R}^3$
Regular boundary	Half-space $\mathbb{R}^3_+$
Edge	Infinite wedge $\mathcal{W}_\alpha := \mathcal{S}_\alpha \times \mathbb{R}$
Corner	3d cone $\mathfrak{C}$

- Ω polyhedral: all the tangent cones are straight (no curvature).
- In general corner domains: the tangent cones have curvature (unbounded). Example: circular cone.

Examples

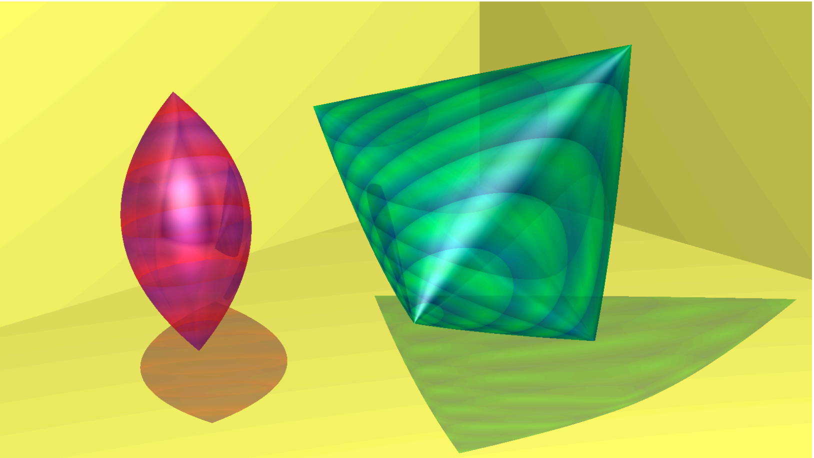

Figure: Domains with conical points

Examples

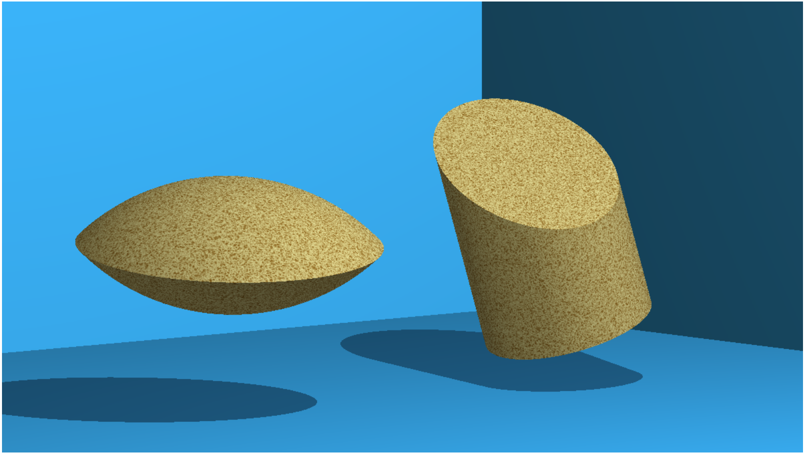

Figure: Domains with edges

Examples

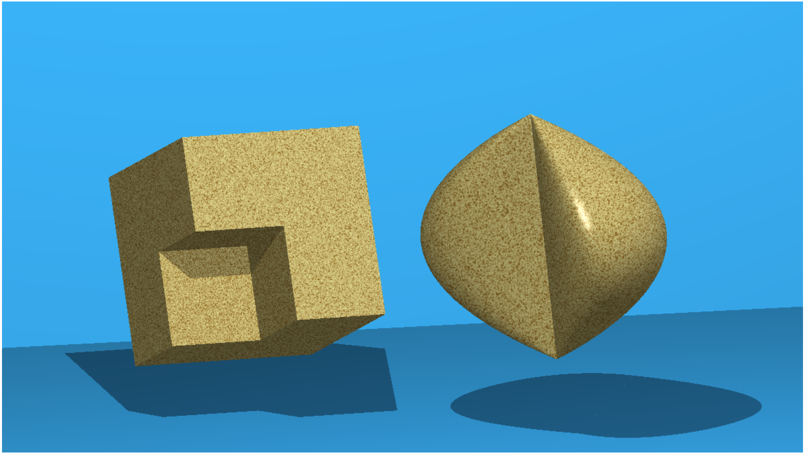

Figure: Domains with corners and edges

Tangent operator and first energy level

- For all $x \in \overline{\Omega}$ define the constant magnetic field $\mathbf{B}_x := \mathbf{B}(x)$. Choose a linear potential $\mathbf{A}_x$ such that $\operatorname{curl} \mathbf{A}_x = \mathbf{B}_x$.

Definition: Tangent operator

We define the tangent operator at $x \in \overline{\Omega}$ as $H(\mathbf{A}_x, \Pi_x)$.

- Scaling: $H(\mathbf{A}_x, \Pi_x)[h] \simeq hH(\mathbf{A}_x, \Pi_x)$.

Definition: Local ground energy

We define the local energy of $x \in \overline{\Omega}$ as

$$E(\mathbf{B}_x, \Pi_x) \quad \text{the ground energy of} \quad H(\mathbf{A}_x, \Pi_x)\,.$$

Examples with unitary magnetic field: ($|\mathbf{B}| = 1$)

- Full space: $E(\mathbf{B}, \mathbb{R}^3) = 1$.
- Half-space: $E(\mathbf{B}, \mathbb{R}^3_+) = \sigma(\theta)$ with θ the angle between $\mathbf{B}$ and $\partial\mathbb{R}^3_+$.
- Wedges $\mathcal{W}_\alpha$ with $\mathbf{B}$ along the edge: $E(\mathbf{B}, \mathcal{W}_\alpha) = E(1, \mathcal{S}_\alpha) = \mu(\alpha)$.

Tangent substructures and second energy level

Definition: Ground energy along higher singular chains

We define

$$\mathscr{E}^*(\mathbf{B}, \Pi_x) := \inf_{\Pi_{\mathbb{X}} \neq \Pi_x} E(\mathbf{B}, \Pi_{\mathbb{X}})$$

where the infimum is taken over tangent substructure.

<u>Example:</u> Take a 3d cone $\mathfrak{C}$ whose section has one vertex of opening α and **B** a constant unitary magnetic field.

Tangent substructures:
- One wedge $\mathcal{W}_\alpha$
- A continuous family of half-spaces Π_θ
- The full space $\mathbb{R}^3$

θ is the angle between a half-space and **B**. Here

$$\mathscr{E}^*(\mathbf{B}, \mathfrak{C}) = \inf(E(\mathbf{B}, \mathcal{W}_\alpha), \inf_\theta \sigma(\theta), 1)$$

Interpretation of the second energy level

- In general

$$E(\mathbf{B},\Pi) \leq \mathscr{E}^*(\mathbf{B},\Pi).$$

Theorem (Essential spectrum on 3d cones) [Bonnaillie-Noël, Dauge, P. 14]

Let $\mathfrak{C}$ be a 3d cone and $\mathbf{B}$ a constant magnetic feld. Then

The bottom of the essential spectrum of $H(\mathbf{A},\mathfrak{C})$ is $\mathscr{E}^*(\mathbf{B},\mathfrak{C})$.

- Consequence: if $E(\mathbf{B},\mathfrak{C}) < \mathscr{E}^*(\mathbf{B},\mathfrak{C})$, we have an eigenfunction with exponential decay for $H(\mathbf{A},\mathfrak{C})$.
- For wedges and half-planes, $\mathscr{E}^*(\mathbf{B},\Pi)$ is a threshold in the spectrum. It is explicit using the function $\theta \mapsto \sigma(\theta)$.

Continuity of the energy

Partition of $\overline{\Omega}$ using *strata* (interior-faces-edges-vertices) :

$$\overline{\Omega} = \Omega \cup \{\mathbf{f}\} \cup \{\mathbf{e}\} \cup \{\mathbf{v}\}$$

- On Ω: $E(\mathbf{B}_x, \Pi_x) = |\mathbf{B}_x|$ is continuous.
- On a face: $E(\mathbf{B}_x, \Pi_x) = |\mathbf{B}_x|\sigma(\theta_x)$ is continuous.

Lemma [P. 13]

The function $(\mathbf{B}, \alpha) \mapsto E(\mathbf{B}, \mathcal{W}_\alpha)$ is $\frac{1}{3}$-Hölder on $\mathbb{S}^2 \times (0, 2\pi)$.

- Define a partial order and a topology on the singular chains.
- See the local energy as a continuous and monotonous function on singular chains:

Theorem [Bonnaillie-Noël, Dauge, P. 14]

The function $x \mapsto E(\mathbf{B}_x, \Pi_x)$ is lower semi-continuous on $\overline{\Omega}$.

- Consequence: $x \mapsto E(\mathbf{B}_x, \Pi_x)$ reaches its infimum over $\overline{\Omega}$.

Convergence in corner domains Ω

- Let $\mathbf{B}$ be a magnetic field with $\inf_\Omega |\mathbf{B}| \neq 0$.
- Let $\boxed{\mathscr{E}(\mathbf{B},\Omega) := \inf_{x\in\overline{\Omega}} E(\mathbf{B}_x,\Pi_x)\,.}$
- Lower semi-continuity and strict diamagnetic inequality: $\mathscr{E}(\mathbf{B},\Omega) > 0$.

Theorem Bonnaillie-Noël, Dauge, P. 14]

Let Ω be a corner domain ($n = 2, 3$) and $\mathbf{B}$ be a regular magnetic field. Let $\mathbf{A}$ be an associated magnetic potential with $\mathbf{A} \in W^{2,\infty}(\Omega)$. Then

$$|\lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega)| \leq C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^\kappa,$$

- Ω polyhedral: $\kappa = 5/4$,
- Ω general: $\kappa = 11/10$.

Proof based on

- Use of minimizer for local energies and generalized eigenfunctions.
- Recursive estimates combined with multiscale analysis.

General estimates in dimension n:

$$\lambda_h(\mathbf{B},\Omega) \geq h\mathscr{E}(\mathbf{B},\Omega) + O(h^{1+1/(3\cdot 2^{n-1}-2)}) \text{ and } \lim_{h\to 0} \frac{\lambda_h(\mathbf{B},\Omega)}{h} = \mathscr{E}(\mathbf{B},\Omega).$$

Plan

Influence of the curvature in 2d

Theorem [Helffer-Morame 01]

Assume that $B = 1$ and $\Omega \subset \mathbb{R}^2$ is regular, with $\kappa_{\max} > 0$ the maximum of the curvature of the boundary. Then there exists $M_0 > 0$:

$$\lambda_h(\mathbf{B}, \Omega) = h\Theta_0 - M_0 \kappa_{\max} h^{3/2} + O(h^{5/3}),$$

The eigenfunctions are localized near the points with maximal curvature.

Similar results for variable magnetic fields, see Raymond [09].

Theorem [Fournais-Helffer 06]

Assume moreover that the the curvature admits a unique non-degerate maximum, then there exists $M_1 > 0$:

$$\lambda_h^k(\mathbf{B}, \Omega) = h\Theta_0 - M_0 \kappa_{\max} h^{3/2} + M_1(2k-1)h^{7/4} + h^{15/8} \sum_{j \geq 0} \gamma_{j,n} h^{j/8}$$

Central tool: Agmon estimates for phase-space localization of the eigenfunctions.

Dimension 3

Theorem [Helffer-Morame 04]

Assume that $\mathbf{B}$ is constant and unitary, and that $\Omega \subset \mathbb{R}^3$ satisfies additional geometrical conditions:

$$\exists \gamma(\Omega, \mathbf{B}) > 0 \quad \lambda_h(\mathbf{B}, \Omega) = h\Theta_0 + \gamma(\Omega, \mathbf{B}) h^{4/3} + O(h^{4/3+\eta}), \ \eta > 0$$

Theorem [P.-Raymond 13]

Assume that $\mathbf{B}$ is constant and Ω is a lens, whose opening admits a unique non degenerate maximum at $\mathbf{v}_0$. Make some hypotheses on the tangent operator at $\mathbf{v}_0$. Then

$$\lambda_h^k(\mathbf{B}, \Omega) = hE(\mathbf{B}, \Pi_{\mathbf{v}_0}) + h^{3/2} \sum_{j \geq 0} \gamma_{j,k} h^{j/2}$$

Corner concentration

Theorem [Generalization of Bonnaillie-Dauge [06]]

Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) be a corner domain such that the local energy reaches its infimum at a corner $\mathbf{v}$ and that the corresponding energy is an isolated eigenvalue: $E(\mathbf{B_v}, \Pi_\mathbf{v}) < \mathscr{E}^*(\mathbf{B_v}, \Pi_\mathbf{v})$. Then there exists K and energies E^k such that

$$\forall 1 \leq k \leq K, \quad \lambda_h^k(\mathbf{B}, \Omega) = hE^k + h^{3/2} \sum_{j \geq 0} \gamma_{j,k} h^{j/2}, \quad E^1 = E(\mathbf{B_v}, \Pi_\mathbf{v}).$$

Moreove, the K first eigenfunctions concentrate near corners of Ω.

2d corner of opening α: $E(1, \mathcal{S}_\alpha) = \mu(\alpha)$ and $\mathscr{E}^*(1, \mathcal{S}_\alpha) = \Theta_0$.

- $\mu(\alpha) \sim \frac{\alpha}{\sqrt{3}}$ as $\alpha \to 0$ ([Bonnaillie 04]).
- $\alpha \mapsto \mu(\alpha)$ increasing is still open!
- $\mu(\alpha) < \Theta_0$ iff $\alpha \in (0, \pi)$ is still open!

Challenge: Sufficient and necessary condition for a 3d cone Π. We focus on sufficient conditions.

A sufficient condition: the sharp cones

Theorem [Bonnaillie-Dauge-P.-Raymond 15]

For a planar bounded domain ω, we define Π_ω the 3d cone whose section along a fixed plane is ω. Define the planar moments

$$m_p := \frac{1}{|\omega|}\int_\omega x_1^p x_2^{2-p}\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad p \in \{0,1,2\}$$

and $N_\omega(\mathbf{B}) = \left(B_3^2 \frac{m_0 m_2 - m_1^2}{m_0+m_2} + B_2^2 m_2 + B_1^2 m_1 - 2B_1 B_2 m_1\right)^{1/2}$. Then

$$\forall \epsilon > 0, \quad E(\mathbf{B}, \Pi_{\epsilon\omega}) \leq \epsilon N_\omega(\mathbf{B}),$$

and $\mathbf{B} \mapsto N_\omega(\mathbf{B})$ is a norm. Moreover, for ϵ small enough, $E(\mathbf{B}, \Pi_{\epsilon\omega})$ is a discrete eigenvalue for the operator $H(\mathbf{A}, \Pi_{\epsilon\omega})$.

Upper bound sharp for a circular cone, see [Bonnaillie-Noël Raymond 14].

Corollary: Corner concentration happens naturally for corner domain with accute vertices.

Plan

The Robin Laplacian

The Robin Laplacian

- The Laplacian with mixed boundary condition $\partial_n u - \alpha u = 0$ on $\partial\Omega$.
- Quadratic form:

$$u \mapsto \int_\Omega |\nabla u|^2 - \alpha \int_{\partial\Omega} |u|^2 \mathrm{d}S \quad u \in H^1(\Omega).$$

- Let $\mu_\alpha(\Omega)$ the bottom of the spectrum
  For Ω bounded let $\mu_\alpha^k(\Omega)$ the k-th eigenvalue.

Recent problematics:

- Clearly $\mu_\alpha(\Omega) \to -\infty$ as $\alpha \to +\infty$. Find refined Asymptotics, depending on Ω.
  It is a semi-classical problem!
- Study the spectral gap as $\alpha \to +\infty$.

Corner domains

Assume that Ω is a corner domain with the uniform interior cone property. Then [Levintin-Parnovski 08]:

$$\mu_\alpha(\Omega) = \alpha^2 \mathscr{E}^R(\Omega) + o(\alpha^2) \quad \text{with} \begin{cases} \mathscr{E}^R(\Omega) = \inf_{x \in \overline{\Omega}} E^R(\Pi_x), \\ E^R(\Pi_x) = \mu_1(\Pi_x) \end{cases}$$

As for the magnetic case! But the energies may be more explicit:

$$E^R(\mathbb{R}^n_+) = -1 \quad \text{and} \quad E^R(\mathcal{S}_\alpha) = \begin{cases} -\sin^{-2}\frac{\alpha}{2} & \text{if } \alpha \in (0, \pi] \\ -1 & \text{if } \alpha \in [\pi, 2\pi) \end{cases}$$

For $\Omega \subset \mathbb{R}^n$ regular: $\mathscr{E}^R(\Omega) = -1$.

- Corner concentrations and tunneling effect in polygons ([Hellfer Pankrashkin 14]).
- Two side estimates for the energy on cones included in a half-space ([Levintin Parnovski 08]).
  This involves that $E^R(\Pi_{\epsilon\omega})$ goes to $-\infty$ for sharp cones ($\epsilon \to 0$).

Influence of the curvature

Let $\Omega \subset \mathbb{R}^2$ regular, then [Pankrashkin 13], [Exner Minakov Parnovski 14]:

$$\mu_\alpha^k(\Omega) = -\alpha^2 - \alpha\kappa_{\max} + O(\alpha^{2/3})$$

Theorem ([Helffer Kachmar 14])

Assume that $\Omega \subset \mathbb{R}^2$ is $\mathcal{C}^\infty$ and that the curvature κ has a unique non degenerate maximum at $s_0 \in \partial\Omega$, then for all $k \geq 1$,

$$\mu_\alpha^k(\Omega) \underset{\alpha\to+\infty}{=} -\alpha^2 - \alpha\kappa_{\max} + (2k-1)\sqrt{\frac{|\kappa''(s_0)|}{2}}\alpha^{1/2} + \sum_{j\geq 0}\gamma_{k,j}\alpha^{-\frac{j}{2}}$$

Theorem [Pankrashkin P. 14]

Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain and denote by H the mean curvature:

$$\forall k \geq 1, \quad \mu_\alpha^k(\Omega) = -\alpha^2 - \max_{\partial\Omega} H\,\alpha + o(\alpha), \quad H_{\max} = \max_{\partial\Omega} H.$$

Reduction to the boundary for the regular case

Theorem [Pankrashkin P. 15]

Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^2$ domain with compact boundary, and H the mean curvature of the boundary. Let $-\Delta^S$ be the Laplace-Beltrami operator on $\partial\Omega$ and

$$\lambda_\alpha^k(\partial\Omega) \text{ the } k\text{-th eigenvalue of the operator } -\Delta^S - \alpha(n-1)H.$$

Then for all $k \geq 1$:

$$\mu_\alpha^k(\Omega) \underset{\alpha \to +\infty}{=} -\alpha^2 + \lambda_\alpha^k(\partial\Omega) + O(\log \alpha).$$

Moreover, if the boundary is C^3, the remainder is improved to $O(1)$.

- The theorem is still valid if $\partial\Omega$ is non compact, provided geometrical assumptions at infinity.

The reduced operator:

- It is semi-classical and the mean curvature acts as a potential.
- Three terms asymptotics in case of “mean curvature wells”.

Plan

Recapitulative of analogies

	Magnetic Laplacian $h \to 0$	Robin Laplacian $\alpha \to \infty$
Equivalent for corner domains	$\lambda_h(\mathbf{B}, \Omega) \sim h\mathscr{E}(\mathbf{B}, \Omega)$ With remainder if $n \leq 3$	$\mu_\alpha(\Omega) \sim \alpha^2 \mathscr{E}^R(\Omega)$
Regularity of local energies	Continuity on strata Global semi-continuity	?
Discrete spectrum for sharp cones $\mathfrak{C}_\epsilon$	Valid for $n = 2, 3$ $E(\mathfrak{C}_\epsilon, \mathbf{B}) \to 0$ as $\epsilon \to 0$	Valid for all n $E^R(\mathfrak{C}_\epsilon) \to -\infty$ as $\epsilon \to 0$
Influence of the curvature in regular cases	Localization in (magnetic) curvature wells ($n \leq 3$)	Localization in mean curvature wells
Global Reduction to the boundary in regular case	?	Effective Hamiltonan: $-\alpha^2 - \Delta^S - \alpha(n-1)H$