The Peierls-Onsager substitution in the framework of *magnetic* ΨDO

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Introduction

The purpose of my talk is to present how the *magnetic ΨDO* calculus developed in collaboration with Viorel Iftimie and Marius Măntoiu, and its equivalent version of *magnetic integral kernels* developed by Horia Cornean and Gheorghe Nenciu allow to obtain a large number of general results concerning the so-called *Peierls-Onsager substitution procedure* for a periodic quantum system in a magnetic field that is not supposed to vanish at infinity. For the moment we consider only the isolated band situation.
The main features of our analysis are that:

- all the formulae and procedures are explicitly gauge covariant;
- no adiabatic hypothesis is needed;
- no decay at infinity is needed for the magnetic field;
- we work only with absolutely convergent expansions and not with asymptotic ones.

Contents

The framework

The framework

- The configuration space: $\mathcal{X} := \mathbb{R}^n$ with a fixed algebraic basis $\{e_1, \ldots, e_d\}$.
- The lattice $\Gamma := \oplus_{j=1}^{d} \mathbb{Z} e_j$, with $\{e_1, \ldots, e_d\}$ an algebraic basis of $\mathcal{X}$. It is isomorphic to $\mathbb{Z}^d$.
- The quotient group $\mathcal{X}/\Gamma$ that is canonically isomorphic to the d-dimensional torus $\mathbb{T}^d \equiv \mathbb{T}$.
- The elementary cell

$$E = \left\{ y = \sum_{j=1}^{d} t_j e_j \in \mathcal{X} \mid -(1/2) \leq t_j < (1/2), \ \forall j \in \{1, \ldots, d\} \right\}.$$

The framework

- The dual space $\mathcal{X}^*$ with the duality map $< .,. >: \mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$.
- The phase space: $\Xi := \mathbb{T}^*\mathcal{X} \cong \mathcal{X} \times \mathcal{X}^*$ with the canonical symplectic form $\sigma^\circ\big((x,\xi),(y,\eta)\big) :=< \xi, y > - < \eta, x >$.
- The dual basis $\{e_1^*, \ldots, e_d^*\} \subset \mathcal{X}^*$ defined by $< e_j^*, e_k >= (2\pi)\delta_{jk}$.
- The dual lattice defined as $\Gamma_* := \{\gamma^* \in \mathcal{X}^* \mid < \gamma^*, \gamma > \in (2\pi)\mathbb{Z},\ \forall \gamma \in \Gamma\}$.
- The dual cell (the Brillouin region):

$$E_* = \Big\{\xi = \sum_{j=1}^{d} t_j e_j^* \in \mathcal{X} \mid -(1/2) \leq t_j < (1/2),\ \forall j \in \{1, \ldots, d\}\Big\}.$$

The Weyl quantization

The Weyl system

Two strongly continuous unitary representations on $\mathcal{H} := L^2(\mathcal{X})$:

$$\mathcal{X} \ni x \mapsto U(x) \in \mathcal{U}(\mathcal{H}), \quad [U(x)f](y) := f(y-x)$$
$$\mathcal{X}' \ni \xi \mapsto V(\xi) \in \mathcal{U}(\mathcal{H}), \quad [V(\xi)f](y) := e^{-i<\xi,y>}f(y)$$

satisfying the Weyl commutation relations:

$$U(x)V(\xi) = e^{i<\xi,x>}\, V(\xi)U(x), \qquad x \in \mathcal{X},\ \xi \in \mathcal{X}'.$$

We define

$$W(x,\xi) := e^{(i/2)<\xi,x>}U(-x)V(\xi).$$

The Weyl quantization - 2

The Weyl operators

$$\mathfrak{Op}(\phi) := (2\pi)^{-d} \int_{\Xi} [\mathcal{F}_{\Xi}^{-} \phi](X)\, W(X)\, dX \ \in \mathbb{B}(\mathcal{H})$$

with $[\mathcal{F}_{\Xi}^{-} \phi](X) := (2\pi)^{-d} \int_{\Xi} e^{i\sigma^{\circ}(X,Y)} \phi(Y)\, dY$.

Explicitly we have

$$[\mathfrak{Op}(\phi) f](x) = \int_{X} dz \int_{X'} d\zeta \; e^{i<\zeta,(x-z)>} \phi\left(\frac{x+z}{2}, \zeta\right) f(z)$$

Hypothesis

For $m \in \mathbb{R}$, $\rho \in [0,1]$
we denote by $S^m_\rho(\mathcal{X})_\Gamma$ the subspace of Γ-periodic symbols in $S^m_\rho(\mathcal{X})$.

Hypothesis I

We fix some *classic Hamiltonian* $h \in S^m_1(\mathcal{X})_\Gamma$ with $m > 0$, elliptic and real.

Theorem

The operator $\mathfrak{Op}(h)$ is essentially self-adjoint and its closure H is lower semi-bounded and has the domain

$$\mathcal{H}^m(\mathcal{X}) := \left\{ u \in L^2(\mathcal{X}) \mid (\mathbf{1} - \Delta)^{m/2} u \in L^2(\mathcal{X}) \right\}.$$

The Bloch-Floquet representation

We consider the linear space

$$\mathscr{F} := \left\{ \hat{F} \in L^2_{\mathrm{loc}}(\mathcal{X} \times \mathcal{X}^*) \mid \tau_\gamma \hat{F} = \sigma_{-\gamma} \hat{F} \ \ \forall \gamma \in \Gamma, \tau_{\gamma^*} \hat{F} = \hat{F} \ \forall \gamma^* \in \Gamma^* \right\}$$

$(\sigma_\gamma(\xi) := e^{-i<\xi,x>}, \ [\tau_\gamma f](x) := f(x+\gamma))$

with the Hilbertian norm

$$\left\| \hat{F} \right\|^2 := \int_E \int_{E_*} \left| \hat{F}(x,\xi) \right|^2 \, d\xi \, dx$$

and the *the Bloch-Floquet* unitary map

$$\mathscr{U}_\Gamma : L^2(\mathcal{X}) \to \mathscr{F}, \quad \left(\mathscr{U}_\Gamma f\right)(x,\xi) = \sum_{\gamma \in \Gamma} \sigma_\gamma(\xi) f(x+\gamma)$$

with its inverse having the explicit form

$$\left(\mathscr{U}_\Gamma^{-1} \hat{F}\right)(x_0+\gamma) = |\mathbb{T}_*|^{-1} \int_{\mathbb{T}_*} \overline{\sigma_\gamma(\theta)} \hat{F}(x_0,\theta) d\theta.$$

The Bloch-Floquet representation - 2

We define for any $\theta \in \mathbb{T}_*$

$$\mathscr{F}_\theta := \left\{ f \in L^2_{\text{loc}}(\mathcal{X}) \mid \tau_\gamma f \right\}$$

with the Hilbertian norm $\|f\|^2_{\mathscr{F}_\theta} = \int_E |f(x)|^2 dx$. Then

$$\mathscr{F} \cong \int_{\mathbb{T}_*}^{\oplus} \mathscr{F}_\theta \, d\theta; \quad \mathscr{U}_\Gamma \mathfrak{Op}(h) \mathscr{U}_\Gamma^{-1} = \int_{\mathbb{T}_*}^{\oplus} \mathfrak{Op}(h)|_{\mathscr{F}_\theta} \, d\theta.$$

Theorem

The operator $\mathfrak{Op}(h)|_{\mathscr{F}_\theta}$ is essentially self-adjoint and its closure $\hat{H}(\theta)$ is lower semi-bounded, has domain

$$\mathscr{F}^m_\theta := \left\{ f \in \mathscr{F}_\theta \mid (\mathbf{1} - \Delta)^{m/2} f \in \mathscr{F}_\theta \right\}$$

and a compact resolvent.

The Bloch structure of the spectrum

There exist a family of continuous functions

$$\mathbb{T}_* \ni \theta \mapsto \lambda_j(\theta) \in \mathbb{R}, \quad \forall j \in \mathbb{N}^*$$

such that $\lambda_j(\theta) \leq \lambda_{j+1}(\theta)$ for any $j \in \mathbb{N}^*$ and any $\theta \in \mathbb{T}_*$ and

$$\sigma\big(\hat{H}(\theta)\big) = \underset{j \in \mathbb{N}^*}{\cup} \{\lambda_j(\theta)\}.$$

Each λ_j is smooth on any region of constant multiplicity.

There exist a family of measurable functions

$$\mathbb{T}_* \ni \theta \mapsto \phi_j(\theta) \in \mathscr{F}_\theta, \quad \forall j \in \mathbb{N}^*$$

such that $\hat{H}(\theta)\phi_j(\theta) = \lambda_j(\theta)\phi_j(\theta)$ for any $j \in \mathbb{N}^*$ and any $\theta \in \mathbb{T}_*$.

The isolated spectral band

The Hypothesis II

There exists a compact interval $I \subset \mathbb{R}$ such that

- $\sigma(H) \cap I =: \sigma_I(H) \neq \emptyset$,
- $\mathrm{dist}(I, \sigma(H) \setminus \sigma_I(H)) =: d_0 > 0$.

Then we denote by $\sigma_\infty(H) := \sigma(H) \setminus \sigma_I(H)$ and notice that there exist $j_0 \in \mathbb{N}^*$ and $N \in \mathbb{N}^*$ such that

$$\sigma_I(H) = \bigcup_{1 \leq j \leq N} \{\lambda_{j_0+j}(\mathbb{T}_*)\}.$$

We denote by $J_I := \{j_0 + 1, \dots, j_0 + N\}$ and by $\mathscr{F}_\theta^I := \mathcal{L}\mathrm{in}\{\phi_j(\theta), j \in J_I\}$,

$$\mathscr{F}^I := \int_{\mathbb{T}_*}^{\oplus} \mathscr{F}_\theta^I d\theta,$$

$$\overset{\circ}{\pi}_j := \mathscr{U}_\Gamma^{-1} \left(\int_{\mathbb{T}_*}^{\oplus} |\phi_j(\theta)\rangle\langle\phi_j(\theta)| d\theta \right) \mathscr{U}_\Gamma, \quad \forall j \in J_I.$$

The *'band dynamics'*

The following equalities are evident:

1. $[H, \overset{\circ}{\pi}_j] = 0, \quad \forall j \in J_I,$
2. $HE_I(H) = \sum_{j \in J_I} \mathfrak{Op}(\lambda_j) \overset{\circ}{\pi}_j$

   where each λ_j is considered as a periodic function of the momenta $\xi \in \mathcal{X}^*$.

Unfortunately, while

$$\mathbb{T}_* \ni \theta \mapsto \sum_{j \in J_I} |\phi_j(\theta)\rangle\langle\phi_j(\theta)| \in \mathbb{B}\big(\mathscr{F}\big)$$

is a smooth rank N projection valued function on the torus,
for each $j \in J_I$ the map

$$\mathbb{T}_* \ni \theta \mapsto |\phi_j(\theta)\rangle\langle\phi_j(\theta)| \in \mathbb{B}\big(\mathscr{F}\big)$$

may be rather singular.

The Wannier basis

Hypothesis III

Under Hypothesis II above, there exist N smooth functions

$$\mathbb{T}_* \ni \theta \mapsto \psi_j(\theta) \in \mathscr{F}^I_\theta, \quad \forall j \in \mathbb{N}^*$$

that form an orthonormal basis in $\mathscr{F}^I_\theta$.

For $j \in J_I$ we denote by $w_j := \mathscr{U}_\Gamma \left(\int_{\mathbb{T}_*}^{\oplus} \psi_j(\theta) d\theta \right) \in L^2(\mathcal{X})$.

Then

1. $\{\mathcal{W}_{\gamma,j} := \tau_\gamma w_j\}_{(\gamma,j)\in\Gamma\times J_I}$ is an orthonormal basis in $\mathcal{H}^I := \mathscr{U}_\Gamma^{-1}\mathscr{F}^I$,
2. $\forall m \in \mathbb{N}$, $\sup_{x\in\mathcal{X}} < x - \gamma >^m |\mathcal{W}_{\gamma,j}(x)| < \infty$, $\forall \gamma \in \Gamma$.

The reduced Hamiltonian

Definition

Under Hypothesis III above we define the $N \times N$ matrix valued function

$$\mathbb{T}_* \ni \theta \mapsto \mu(\theta) \in \mathcal{M}_{N,N}(\mathbb{C}), \quad \mu_{jk}(\theta) := \langle \psi_j(\theta), H\psi_k(\theta) \rangle_{\mathscr{F}_\theta}$$

If Hypothesis I, II and III are true, the following statements are evident:

1. $\langle \mathcal{W}_{\alpha,j}, H\mathcal{W}_{\beta,k} \rangle_{L^2(\mathcal{X})} = \widehat{\mu_{jk}}(\alpha - \beta) := |E_*|^{-1} \int_{\mathbb{T}_*} e^{-i<\theta,\alpha-\beta>} \mu_{jk}(\theta) d\theta$,
2. the band Hamiltonian $HE_I(H)$ is unitary equivalent with the following Γ-translation invariant matrix valued operator acting on $l^2(\Gamma) \otimes \mathbb{C}^N$:

$$\big(\mathfrak{M}(h)_{\alpha,\beta}\big)_{jk} := \widehat{\mu_{jk}}(\alpha - \beta).$$

Conclusion

- $[H, \mathring{\pi}_j] = 0$,
- $HE_I(H) = \sum_{j \in J_I} \mathfrak{Op}(\lambda_j)\mathring{\pi}_j$,
- $\langle \mathcal{W}_{\alpha,j}, H\mathcal{W}_{\beta,k} \rangle_{L^2(\mathcal{X})} = \widehat{\mu_{jk}}(\alpha - \beta)$.

The isolated spectral band in a magnetic field

The magnetic field

- The magnetic field is described by a closed 2-form B on $\mathcal{X}$:

$$B : \mathcal{X} \to \mathcal{X} \wedge \mathcal{X}, \qquad dB = 0.$$

- To B we may associate in a highly non-unique way a vector potential, i.e. a 1-form A such that $B = dA$.
- Gauge transformations: $A \mapsto A' = A + d\Phi$; so that $B = dA = dA'$.
- Using the transversal gauge we can define a vector potential A_ϵ such that $B_\epsilon = dA_\epsilon$:

$$A_{\epsilon,j}(x) \; := \; -\sum_{1 \leq k \leq d} x_k \int_0^1 B_{\epsilon,jk}(sx) s ds.$$

Hypothesis on the magnetic field

Hypothesis IV

We consider a magnetic field of the form $B_\epsilon := \epsilon \overset{\circ}{B}_\epsilon$ with $\epsilon \in [0, \epsilon_0]$ for some small enough $\epsilon_0 > 0$ and with a family of magnetic fields $\{\overset{\circ}{B}_\epsilon\}_{\epsilon \in [0,\epsilon_0]}$ having components in a bounded subset of $BC^\infty(\mathcal{X})$
(we implicitly assume that $d\overset{\circ}{B}_\epsilon = 0$ for any $\epsilon \in [0, \epsilon_0]$).

We shall fix a family of vector potentials $\{\overset{\circ}{A}_\epsilon\}_{\epsilon \in [0,\epsilon_0]}$ having components in a bounded subset of $C^\infty_{\text{pol}}(\mathcal{X})$ such that $\overset{\circ}{B}_\epsilon = d\overset{\circ}{A}_\epsilon$;

then $B_\epsilon = dA_\epsilon$ for $A_\epsilon := \epsilon \overset{\circ}{A}_\epsilon$.
We emphasize that all our constructions will be explicitly gauge covariant.

The magnetic Pseudodifferential Calculus

Definition

For any Schwartz test function $\Phi \in \mathscr{S}(\Xi)$ the following oscillating integral defines a continuous linear operator in $\mathscr{S}(\mathcal{X})$:

$$\big(\mathfrak{Op}^A(\Phi)f\big)(x) := (2\pi)^{-n/2}\int_\Xi e^{i<\eta,x-y>}\Lambda^A(x,y)\Phi\Big(\frac{x+y}{2},\eta\Big)f(y)dy\,d\eta$$

$\Lambda^A(x,y) := \exp\big\{-i\int_{[x,y]}A\big\}$.

Gauge covariance

$$A' = A + d\varphi \quad \Rightarrow \quad \mathfrak{Op}^{A'}(f) = e^{i\varphi(Q)}\mathfrak{Op}^A(f)e^{-i\varphi(Q)}.$$

Note that

1. $\Lambda^A(x,y)\Lambda^A(y,z) = \Lambda^A(x,z)\exp\Big\{-i\int_{<x,y,z>}B\Big\} \equiv \Lambda^A(x,z)\Omega^B(x,y,z)$,
2. $|\Omega^B(x,y,z)-1| \leq C\|B\|_\infty|(y-x)\wedge((z-x)|$.

The magnetic Pseudodifferential Calculus - 2

Proposition. MP04

The map $\mathfrak{Op}^A : \mathscr{S}(\Xi) \to \mathbb{B}\big(\mathscr{S}(\mathcal{X})\big)$
is an isomorphism of linear topological spaces that extends to an isomorphism $\mathfrak{Op}^A : \mathscr{S}'(\Xi) \to \mathbb{B}\big(\mathscr{S}(\mathcal{X}); \mathscr{S}'(\mathcal{X})\big)$.

Proposition. IMP07

Under our Hypothesis IV, for any symbol $F \in S^0_0(\mathcal{X})_\Gamma$ we have that $\mathfrak{Op}^A(F) \in \mathbb{B}\big(L^2(\mathcal{X})\big)$
and there exist two constants $C > 0$ and $p \in \mathbb{N}^*$, depending only on the dimension $d \geq 2$ of the configuration space, such that:

$$\|\mathfrak{Op}^A(F)\|_{\mathbb{B}(L^2(\mathcal{X}))} \leq C \sup_{(x,\xi)\in\Xi} \sup_{|a|\leq p} \sup_{|b|\leq p} \big|\big(\partial^a_x \partial^b_\xi F\big)(x,\xi)\big|.$$

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The magnetic Pseudodifferential Calculus - 3

Definition. MP04

On $\mathscr{S}(\Xi)$ we define the following separately continuous bilinear form, that is gauge independent:

$$\mathscr{S}(\Xi)\times\mathscr{S}(\Xi)\ni(f,g)\mapsto f\sharp^B g\in\mathscr{S}(\Xi),\quad \mathfrak{Op}^A(f\sharp^B g):=\mathfrak{Op}^A(f)\mathfrak{Op}^A(g).$$

We have the explicit formula:

$$(f\sharp^B g)(X)=(2\pi)^{-2d}\int_\Xi\int_\Xi e^{-2i\sigma^\circ(Y,Z)}e^{-i\int_{T(x,y,z)}B}f(X-Y)g(X-Z)\,dYdZ,$$

with $T(x,y,z)$ having the vertices $x-y-z, x+y-z, x-y+z$.

The magnetic Integral Kernels Calculus (Cornean, Nenciu)

For any symbol $F \in \mathscr{S}'(\Xi)$ the Weyl operator $\mathfrak{Op}(F)$ has as distribution kernel $K_F \in \mathscr{S}(\mathcal{X} \times \mathcal{X})$ given by

$$K_F(x,y) = \big(\mathfrak{W}F\big)(x,y) := (2\pi)^{-d} \int_{\mathcal{X}^*} e^{i<\xi,(x-y)>} F\big((x+y)/2, \xi\big) d\xi.$$

Proposition

The operator $\mathfrak{Op}^A(F)$ has the distribution kernel:

$$K_F^A(x,y) = \Lambda^A(x,y)\big(\mathfrak{W}F\big)(x,y) = \Lambda^A(x,y) K_F(x,y).$$

The magnetic Pseudodifferential Calculus - 4

Theorem. IMP07

Under Hypothesis I and IV, $\mathfrak{Op}^A(h)$ is essentially self-adjoint and its closure H^A is lower semi-bounded and has domain

$$\mathcal{H}^A(\mathcal{X}) := \Big\{ u \in L^2(\mathcal{X}) \mid (-i\partial - A)^a u \in L^2(\mathcal{X}), \forall a \in \mathbb{N}^d, |a| \leq m \Big\}.$$

Remark. MP04

For $h_0(x,\xi) := \xi^2 + V(x)$, we have

$$\mathfrak{Op}^A(h_0) = \sum_{1 \leq j \leq d} \big(-i\partial_j - A_j(x) \big)^2 + V(Q) \equiv -\Delta_A^2 + V(Q).$$

Theorem. IMP10

Under Hypothesis I and IV, if $\mathfrak{z} \ni \sigma(H^A)$, the resolvent in $\mathfrak{z}$ is given by $\big(H^A - \mathfrak{z}\big)^{-1} = \mathfrak{Op}^A(r^B_{\mathfrak{z}}(h))$ with $r^B_{\mathfrak{z}}(h) \in S^{-m}_1(\mathcal{X})$ depending on the magnetic field but not on the choice of gauge.

The magnetic Pseudodifferential Calculus - 5

Development of the magnetic product

If $f \in S^{m_1}_\rho(\mathcal{X})$ and $g \in S^{m_2}_\rho(\mathcal{X})$,
then for any $n \in \mathbb{N}^*$ there exist the family of symbols
$C^\epsilon_k(f,g) \in S^{m_1+m_2-2k\rho}_\rho(\mathcal{X})$, for $(k,\epsilon) \in \mathbb{N} \times [0,\epsilon_0]$ and
$R^\epsilon_n(f,g) \in S^{m_1+m_2-2n\rho}_\rho(\mathcal{X})$, for $(n,\epsilon) \in \mathbb{N}$,
uniformly with respect to $\epsilon \in [0,\epsilon_0]$, such that

$$f \sharp^\epsilon g = f \sharp^0 g + \sum_{1 \leq k \leq n-1} \epsilon^k C^\epsilon_k(f,g) + \epsilon^n R^\epsilon_n(f,g).$$

The magnetic Pseudodifferential Calculus - 6

Development of the magnetic resolvent

1. [IP15, CP12, CP15]There exists $\epsilon_0 > 0$ small enough such that the bounded interval $I \subset \mathbb{R}$ in Hypothesis II satisfies the two conditions in the Hypothesis for H^ϵ with $\epsilon \in [0, \epsilon_0]$.
2. For any compact set $K \subset \mathbb{C} \setminus \sigma(H)$ there exists $\epsilon_0 > 0$ such that $K \subset \mathbb{C} \setminus \sigma(H^\epsilon)$ for $\epsilon \in [0, \epsilon_0]$ and the function
$$K \ni \mathfrak{z} \mapsto r^\epsilon_{\mathfrak{z}}(h) \in S^{-m}_1(\mathcal{X})$$
is continuous for the Fréchet topology uniformly for $\epsilon \in [0, \epsilon_0]$. Moreover we have the following development
$$r^\epsilon_{\mathfrak{z}}(h) = r^0_{\mathfrak{z}}(h) + \sum_{k \in \mathbb{N}^*} \epsilon^k r_k(h; \epsilon, \mathfrak{z}),$$
with $r_k(h; \epsilon, \mathfrak{z}) \in S^{-(m+2n)}_1(\mathcal{X})$ uniformly for $\epsilon \in [0, \epsilon_0]$ and the series converging in the topology induced from $\mathbb{B}(L^2(\mathcal{X}))$ by the map $\mathfrak{Op}^\epsilon$ for any $\epsilon \in [0, \epsilon_0]$.

The main results

Theorem A

Under Hypothesis I, II and IV, there exists $\epsilon_0 > 0$ small enough such that for any $\epsilon \in [0, \epsilon_0]$ and for any $n \in \mathbb{N}^*$ we have that:

1. $H^\epsilon E_I(H^\epsilon) = \mathfrak{Op}^\epsilon(h_I) + \sum_{1 \le k \le n-1} \epsilon^k \mathfrak{Op}^\epsilon(v_k^\epsilon) + \epsilon^n \mathfrak{Op}^\epsilon(R_I^\epsilon(h; n))$ with:
    1. $h_I(x, \xi) :=$
    $\frac{(2\pi)^d}{|E_*|} \int_{\mathcal{X}} e^{-i<\xi,y>} \left[\int_{\mathbb{T}_*} \left(\sum_{j \in JI} \lambda_j(\theta) \phi_j(x + y/2, \theta) \overline{\phi_j(x - y/2, \theta)} \right) d\theta \right] dy,$
    2. $v_k^\epsilon := -(2\pi i)^{-1} \int_{\mathcal{C}} \mathfrak{z} r_k(h; \epsilon, \mathfrak{z}) d\mathfrak{z},$
    3. $R_I^\epsilon(h; n) := -(2\pi i)^{-1} \int_{\mathcal{C}} \mathfrak{z} \left(\sum_{k \ge n} \epsilon^k r_k(h; \epsilon, \mathfrak{z}) \right) d\mathfrak{z}.$
2. there exists an orthogonal projection $P_{I,n}^\epsilon$ in $L^2(\mathcal{X})$ such that
    1. $\|E_I(H^\epsilon) - P_{I,n}^\epsilon\| \le C_n \epsilon^n, \quad \|H^\epsilon[E_I(H^\epsilon) - P_{I,n}^\epsilon]\| \le C_n(h) \epsilon^n,$
    2. $\|[H^\epsilon, P_{I,n}^\epsilon]\| \le C_n(h) \epsilon^n,$
    3. for $P_{I,n}^\epsilon$ we have a development in powers of $\epsilon \in [0, \epsilon_0]$ similar to the one of $H^\epsilon E_I(H^\epsilon)$ at point 1.

Corollary

For a N-fold degenerated isolated spectral band $\lambda : \mathbb{T}_* \to \mathbb{R}$ we have that

$$H^\epsilon E_I(H^\epsilon) = \mathfrak{Op}^\epsilon(\lambda) E_I(H^\epsilon) + \sum_{1 \leq k \leq n-1} \epsilon^k \mathfrak{Op}^\epsilon\big(v_k^\epsilon\big) + \epsilon^n \mathfrak{Op}^\epsilon\big(R_I^\epsilon(h; n)\big).$$

Theorem B

Under Hypothesis I, II, III and IV, there exists an orthonormal *magnetic Wannier* basis $\{\mathcal{W}^{\epsilon}_{\gamma,j}\}_{(\gamma,j)\in\Gamma\times J_I}$ such that:

1. $E_I(H^{\epsilon}) = \sum_{\gamma\in\Gamma}\sum_{j\in J_I} |\mathcal{W}^{\epsilon}_{\gamma,j}\rangle\langle\mathcal{W}^{\epsilon}_{\gamma,j}|$,
2. $\sup_{x\in\mathcal{X}} < x-\gamma >^m |\mathcal{W}^{\epsilon}_{\gamma,j}| < \infty,\ \forall(\gamma,j)\in\Gamma\times J_I$,
3. $\langle\mathcal{W}^{\epsilon}_{\alpha,j}, H^{\epsilon}\mathcal{W}^{\epsilon}_{\beta,k}\rangle_{L^2(\mathcal{X})} = \Lambda^{\epsilon}(\alpha,\beta)\widehat{\mu_{jk}}(\alpha-\beta) + C\epsilon$

for any $(\alpha,\beta)\in\Gamma\times\Gamma$, $(j,k)\in J_I\times J_I$ and $\epsilon\in[0,\epsilon_0]$.

The case of constant magnetic field I.

- We consider satisfied Hypothesis I and II and a constant magnetic field B.
- We define the unitary mapping
  $\mathfrak{R}_\Gamma : L^2(\mathcal{X}) \to l^2\big(\Gamma; L^2(E)\big) \cong l^2(\Gamma) \otimes L^2(E)$

  given by the formula $\big(\mathfrak{R}_\Gamma f\big)_\gamma(\hat{x}) := f(\gamma + \hat{x})$.
- We denote by $\overset{\circ}{\tau} : \Gamma \to \mathbb{B}\big(l^2(\Gamma)\big)$ the restriction of the translations to $l^2(\Gamma)$.
- We consider the unitary transformation in $L^2(\mathcal{X})$ defined by $\big(\Upsilon^A f\big)(x) := \Lambda^A([x], x) f(x)$.

The case of constant magnetic field I.

Theorem I

Under the above assumptions we have that

$$H^A E_I(H^A) = \left(\Upsilon^A \mathfrak{R}_\Gamma\right)^{-1} \mathfrak{M}^B(h) \left(\Upsilon^A \mathfrak{R}_\Gamma\right)$$

with

$$\mathfrak{M}^B(h)_{\alpha,\beta} := \Lambda^A(\alpha,\beta) \mathfrak{H}_I^B(\alpha-\beta)$$

where $\mathfrak{H}_I^B(\gamma) \in \mathbb{B}\big(L^2(E)\big)$ is defined by the following integral kernel:

$$k^B_{h,I}(\gamma,\hat{x},\hat{y}) := \Phi^B_\gamma(\hat{x},\hat{y}) k_{h,I}(\gamma,\hat{x},\hat{y})$$

$$\Phi^B_\gamma(\hat{x},\hat{y}) \;=\; \exp\left\{(-i/2)\big(\langle B,\gamma\wedge(\hat{x}+\hat{y})\rangle + \langle B,\hat{x}\wedge\hat{y}\rangle\big)\right\},$$

$$k_{h,I}(\gamma,\hat{x},\hat{y}) := \int_{\mathbb{T}_*} e^{-i<\theta,\gamma>}\left(\sum_{j\in J_I}\lambda_j(\theta)\phi_j(\hat{x}+\hat{y}/2,\theta)\overline{\phi_j(\hat{x}-\hat{y}/2,\theta)}\right) d\theta.$$

The case of constant magnetic field II.

Theorem II

Under the above assumptions in the transverse gauge representation, we have the following convolution form for the matrix of the band Hamiltonian in the modified magnetic Wannier basis:

$$\left\langle \mathcal{W}^{\epsilon}_{\alpha,l}, H^{\epsilon}\mathcal{W}^{\epsilon}_{\beta,m}\right\rangle_{L^2(\mathcal{X})} = \Lambda^{\epsilon}(\alpha,\beta)\left\langle \mathcal{W}^{\epsilon}_{\alpha-\beta,l}, H^{\epsilon}\mathcal{W}^{\epsilon}_{0,m}\right\rangle_{L^2(\mathcal{X})} \equiv \widetilde{\Lambda}^{\epsilon}(\alpha,\beta)\hat{\mathfrak{H}}^{\epsilon}_{lm}(\alpha-\beta).$$

Thank you !