Large deviations for Markov-modulated diffusion processes with rapid switching

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Outline

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The model

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, where $\mathbb{R}_+ := [0, +\infty)$. $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$, and $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is right continuous.

$X_t, t \geq 0$ is a finite-state time-homogeneous Markov chain with transition intensity matrix Q and state space $\mathbb{S} := \{1, \cdots, d\}$.

The *Markov-modulated diffusion process* is the unique solution to

$$M_t = M_0 + \int_0^t b(X_s, M_s)\mathrm{d}s + \int_0^t \sigma(X_s, M_s)\mathrm{d}B_s,$$

where B is a standard Brownian motion. See assumptions.

Assumptions

(A.1) *Lipschitz continuity*: there is $K > 0$ such that $\forall i \in \mathbb{S}, x, y \in \mathbb{R}$

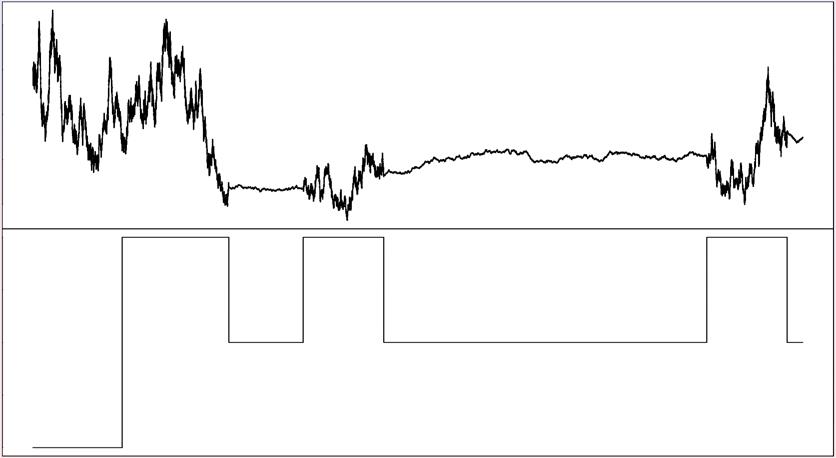
$$|b(i,x) - b(i,y)| + |\sigma(i,x) - \sigma(i,y)| \leq K|x-y|, \quad .$$

(A.2) *Linear growth*: there $K > 0$ such that

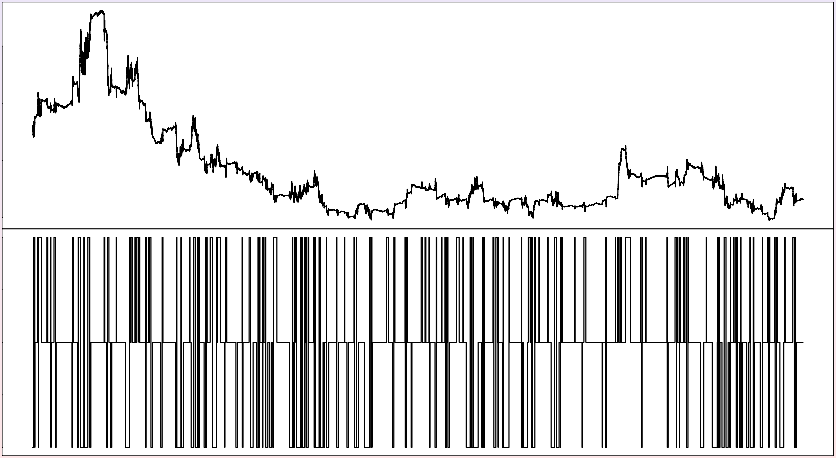
$$|b(i,x)| + |\sigma(i,x)| \leq K(1+|x|), \quad \forall i \in \mathbb{S}, \quad x \in \mathbb{R}.$$

(A.3) *Irreducibility*: the Markov chain X_t is irreducible and has an invariant probability measure $\pi = (\pi(1), \cdots, \pi(d))$.

MM diffusion with slowly jumping chain



MM diffusion with faster jumping chain



Objective: large deviations principle for $\epsilon \to 0$ (LDP)

- Study the above SDE under scaling: Scale Q to $Q/\epsilon =: Q^\epsilon$; X_t^ϵ is the Markov chain with transition intensity matrix Q^ϵ.
- At the same time small-noise large deviations (Freidlin and Wentzell [4]). Scaling of the function $\sigma(\cdot,\cdot)$ to $\sqrt{\epsilon}\sigma(\cdot,\cdot)$. The resulting process M_t^ϵ is the unique strong solution to

$$M_t^\epsilon = M_0^\epsilon + \int_0^t b(X_s^\epsilon, M_s^\epsilon)\mathrm{d}s + \sqrt{\epsilon}\int_0^t \sigma(X_s^\epsilon, M_s^\epsilon)\mathrm{d}B_s.$$

  We assume $M_0^\epsilon \equiv 0$, whereas X_0^ϵ starts at an arbitrary $x \in \mathbb{S}$.
- Investigate the *LDP* for the *joint* process $(M^\epsilon, \nu^\epsilon)$, where

$$\nu^\epsilon(\omega; t, i) = \int_0^t \mathbf{1}_{\{X_s^\epsilon(\omega)=i\}}\mathrm{d}s.$$

Additional notions

- $\mathbb{M}_T$ is the space of functions ν on $[0, T] \times \mathbb{S}$ satisfying $\nu(t,i) = \int_0^t K_\nu(s,i)\mathrm{d}s$, where $\sum_{i=1}^d K_\nu(s,i) = 1$, $K_\nu(s,i) \geq 0$, and $K_\nu(\cdot, i)$ Borel measurable. The metric on $\mathbb{M}_T$ is

$$d_T(\mu,\nu) = \sup_{0 \leq t \leq T, i \in \mathbb{S}} \left| \int_0^t K_\mu(s,i)\mathrm{d}s - \int_0^t K_\nu(s,i)\mathrm{d}s \right|.$$

- $\mathbb{C}_T = \{f \in \mathbb{C}_{[0,T]}(\mathbb{R}) : f(0) = 0\}$ with the uniform metric ρ_T.
- The product metric $\rho_T \times d_T$ on $\mathbb{C}_T \times \mathbb{M}_T$ is defined by

$$(\rho_T \times d_T)((\varphi,\nu),(\varphi',\nu')) := \rho_T(\varphi,\varphi') + d_T(\nu,\nu').$$

$\mathcal{B}(\mathbb{C}_T \times \mathbb{M}_T)$ is the Borel σ-algebra generated by $\rho_T \times d_T$.

Outline

Large deviations principle (LDP)

Let $\mathbb{X}$ be a Polish space with metric ρ and Borel σ-algebra $\mathcal{B}(\mathbb{X})$.

Definition 1 (Varadhan [8])

A family of probability measures $\mathbb{P}^\epsilon$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is said to obey the LDP with a rate function $I(\cdot) : \mathbb{X} \to [0, \infty]$ satisfying:

1. There exists $x \in \mathbb{X}$ such that $I(x) < \infty$; I is lsc; for every $c < \infty$ the set $\{x : I(x) \leq c\}$ is compact in $\mathbb{X}$.
2. For every closed set $F \subset \mathbb{X}$,
   $\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}^\epsilon(F) \leq -\inf_{x \in F} I(x)$.
3. For every open set $O \subset \mathbb{X}$,
   $\liminf_{\epsilon \to 0} \epsilon \log \mathbb{P}^\epsilon(O) \geq -\inf_{x \in O} I(x)$.

Exponential tightness

Definition 2 (Den Hollander [3], Puhalskii [7])

A family of probability measures $\mathbb{P}^\epsilon$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is said to be exponentially tight, if for every $L < \infty$, there exists a compact set $K_L \subset \mathbb{X}$ such that

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}^\epsilon(\mathbb{X} \setminus K_L) \leq -L.$$

Local LDP

Definition 3 (Puhalskii [7], Liptser and Puhalskii [5])

A family of probability measures $\mathbb{P}^\epsilon$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is said to obey the local LDP with a rate function $I(\cdot)$ if for every $x \in \mathbb{X}$

$$\limsup_{\delta \to 0} \limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}^\epsilon(\{y \in \mathbb{X} : \rho(x, y) \leq \delta\}) \leq -I(x), \tag{1}$$

$$\liminf_{\delta \to 0} \liminf_{\epsilon \to 0} \epsilon \log \mathbb{P}^\epsilon(\{y \in \mathbb{X} : \rho(x, y) \leq \delta\}) \geq -I(x). \tag{2}$$

LDP and local LDP

Since $\mathbb{X}$ is a Polish space, Definition 1(1) implies exponential tightness. Definition 1(2,3) guarantee that $\mathbb{P}^{\epsilon}$ satisfies the local LDP. The converse is also valid and is the key for our main result.

Theorem 4 (Puhalskii [7], Liptser and Puhalskii [5])

If a family of probability measures $\mathbb{P}^{\epsilon}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is exponentially tight and obeys the local LDP with a rate function I, then it obeys the LDP with the rate function I.

LDP on a dense subset

A local LDP on a dense subset of $\mathbb{X}$ implies the local LDP on $\mathbb{X}$.

Lemma 5 (Borovkov and Mogulskiĭ [2])

(i) If (1) is fulfilled for all $\tilde{x} \in \tilde{\mathbb{X}}$, where $\tilde{\mathbb{X}}$ is dense in $\mathbb{X}$ and function $I(x)$ is lower semi-continuous, then it holds for all $x \in \mathbb{X}$.
(ii) If for every $x \in \mathbb{X}$ with $I(x) < \infty$ there exists a sequence $\tilde{x}_n \in \tilde{\mathbb{X}}$ converging to x and $I(\tilde{x}_n) \to I(x)$, then (2) for $\tilde{x} \in \tilde{\mathbb{X}}$ implies the same for all $x \in \mathbb{X}$.

Outline

Rate function for the Markov chain

The rate function corresponding to ν^ϵ is defined as

$$\tilde{I}_T(\nu) := \int_0^T \sup_{u \in U} \left[- \sum_{i=1}^{d} \frac{(Qu)(i)}{u(i)} K_\nu(s,i) \right] \mathrm{d}s, \quad \nu \in \mathbb{M}_T,$$

where

$(Qu)(i) = \sum_{j=1}^{d} Q_{ij} u(j)$, for $i \in \mathbb{S}$, $U = \mathbb{R}^d_{++}$.

NB: $\tilde{I}_T(\nu)$ is a time varying variation on the usual rate function for large deviations of Markov chains [3, Theorem IV.14].

Rate function for M^ϵ

Let $\mathbb{H}_T = \{\varphi \in \mathbb{C}_T : \varphi(t) = \int_0^t \varphi'(s)\mathrm{d}s, \text{ with } \varphi' \in L^2[0,T]\}$ (Cameron-Martin space).

The rate function corresponding to M^ϵ is

$$I_T(\varphi,\nu) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \frac{[\varphi'_t - \hat{b}_t(\nu,\varphi_t)]^2}{\hat{\sigma}_t^2(\nu,\varphi_t)}\mathrm{d}t & \text{if } \varphi \in \mathbb{H}_T, \\ \infty & \text{otherwise.} \end{cases}$$

where

$$\hat{b}_t(\nu,x) := \sum_{i=1}^d b(i,x)K_\nu(t,i)$$

$$\hat{\sigma}_t(\nu,x) := \left(\sum_{i=1}^d \sigma^2(i,x)K_\nu(t,i)\right)^{1/2}.$$

Main theorem

Let $\mathbb{P} \circ (M^\epsilon, \nu^\epsilon)^{-1}$ denote $\mathbb{P}((M^\epsilon, \nu^\epsilon) \in \cdot)$, a family of probability measures on $(\mathbb{C}_T \times \mathbb{M}_T, \mathcal{B}(\mathbb{C}_T \times \mathbb{M}_T))$.
The marginals $\mathbb{P} \circ (M^\epsilon)^{-1}$ and $\mathbb{P} \circ (\nu^\epsilon)^{-1}$ are families of probability measures on $(\mathbb{C}_T, \mathcal{B}(\mathbb{C}_T))$ and $(\mathbb{M}_T, \mathcal{B}(\mathbb{M}_T))$ respectively.

Theorem 6

For every $T > 0$, the family $\mathbb{P} \circ (M^\epsilon, \nu^\epsilon)^{-1}$ obeys the LDP in $(\mathbb{C}_T \times \mathbb{M}_T, \rho_T \times d_T)$ with the rate function

$$L_T(\varphi, \nu) = I_T(\varphi, \nu) + \tilde{I}_T(\nu).$$

Two corollaries

The following two results are a consequence of the *contraction principle*.

Corollary 7

The family $\mathbb{P} \circ (M^\epsilon)^{-1}$ *obeys the LDP with the rate function* $\inf_{\nu \in \mathbb{M}_T} L_T(\varphi, \nu)$.

Corollary 8

The family $\mathbb{P} \circ (\nu^\epsilon)^{-1}$ *obeys the LDP in* $(\mathbb{M}_T, d_T)$ *with the rate function* $\tilde{I}_T(\nu)$.

Outline

Structure of the proof of the main theorem I

Prove exponential tightness of $\mathbb{P} \circ (M^\epsilon, \nu^\epsilon)^{-1}$, i.e., for every $L < \infty$, there exists a compact set $K_L \subset \mathbb{C}_T \times \mathbb{M}_T$ such that

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}\left((M^\epsilon, \nu^\epsilon) \in \mathbb{C}_T \times \mathbb{M}_T \setminus K_L\right) \leq -L.$$

Steps:

- $\mathbb{P} \circ (M^\epsilon, \nu^\epsilon)^{-1}$ is exponentially tight if $\mathbb{P} \circ (M^\epsilon)^{-1}$ and $\mathbb{P} \circ (\nu^\epsilon)^{-1}$ are so.
- Exponential tightness of $\mathbb{P} \circ (M^\epsilon)^{-1}$ in Proposition 9 below.
- For any $\nu \in \mathbb{M}_T$, its derivative $K_\nu(s,i)$ is bounded by 1, hence $\mathbb{M}_T$ is equicontinuous. Moreover, $\mathbb{M}_T$ is bounded and closed and the Arzelà-Ascoli theorem implies that $\mathbb{M}_T$ is compact. We can take $K_L = \mathbb{M}_T$ for $\mathbb{P} \circ (\nu^\epsilon)^{-1}$.

Structure of the proof of the main theorem II

Show that $\mathbb{P} \circ (M^\epsilon, \nu^\epsilon)^{-1}$ obeys the local LDP with rate function $L_T(\varphi, \nu)$: for every $(\varphi, \nu) \in \mathbb{C}_T \times \mathbb{M}_T$,

the upper bound

$$\limsup_{\delta \to 0} \limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}(\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \leq \delta) \leq -L_T(\varphi, \nu),$$

and the lower bound

$$\liminf_{\delta \to 0} \liminf_{\epsilon \to 0} \epsilon \log \mathbb{P}(\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \leq \delta) \geq -L_T(\varphi, \nu).$$

Structure of the proof of the main theorem III

Steps for proving the local LDP:

- Prove the local LDP on a dense subset of $\mathbb{C}_T \times \mathbb{M}_T$.
- Prove the upper bound: Proposition 17.
- The lower bound is first proved in Proposition 20 under the condition $\inf_{i,x} \sigma^2(i,x) > 0$.
- Then the condition is lifted in Proposition 22 by a perturbation argument.

Outline

Exponential tightness of $\mathbb{P} \circ (M^\epsilon)^{-1}$

Proposition 9

For every $T > 0$, the family $\mathbb{P} \circ (M^\epsilon)^{-1}$ is exponentially tight on $(\mathbb{C}_T, \mathcal{B}(\mathbb{C}_T))$.

The technique to prove the proposition borrows elements from Liptser [5]. We also use two auxiliary results adapted from Aldous and from Liptser and Pukhalskii [6], applied to $Y = M^\epsilon$.

Auxiliary result I

Let Γ_T be the family of $\mathcal{F}_t$-stopping times with values in $[0, T]$.

Proposition 10 (Aldous [1])

Let, for each $\epsilon > 0$, $Y^\epsilon : \Omega \times [0, T] \to \mathbb{R}$ be an $\{\mathcal{F}_t\}_{t \le T}$-adapted continuous process, so with paths in $\mathbb{C}_T$. If

$$(i) \lim_{K' \to \infty} \limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(Y_T^{\epsilon *} \ge K' \right) = -\infty,$$

and

$$(ii) \lim_{\delta \to 0} \limsup_{\epsilon \to 0} \epsilon \log \sup_{\tau \in \Gamma_T} \mathbb{P}\left(\sup_{t \le \delta} |Y_{\tau+t}^\epsilon - Y_\tau^\epsilon| \ge \eta \right) = -\infty, \forall \eta > 0,$$

then $\mathbb{P} \circ (Y^\epsilon)^{-1}$ is exponentially tight.

Auxiliary result II, needed for (ii)

Lemma 11 (Liptser and Pukhalskii [6])

Let $Y = (Y_t)_{t \geq 0}$ be a continuous semimartingale with $Y_0 = 0$, $Y = A + M$, A is the predictable process of locally bounded variation, M the local martingale.

Assume that for $T > 0$ there exists a convex function H with $H(0) = 0$ and a nonnegative random variable ξ such that for all $\lambda \in \mathbb{R}$ and $t \leq T$

$$\lambda A_t + \lambda^2 \langle M \rangle_t / 2 \leq tH(\lambda \xi), \text{ a.s.}.$$

Then, for all $c > 0$ and $\eta > 0$,

$$\mathbb{P}(Y_T^* \geq \eta) \leq \mathbb{P}(\xi > c) + \exp\left\{ - \sup_{\lambda \in R} [\lambda \eta - TH(\lambda c)] \right\}.$$

Outline

Auxiliary result III

Let $\mathbb{M}_T^+$ be the subset of $\mathbb{M}_T$ such that $K_\nu(s,i) > 0$, and $\mathbb{M}_T^{++}$ be the subset of $\mathbb{M}_T^+$ such that $K_\nu(\cdot,i) \in \mathbb{C}_{[0,T]}^\infty, \forall i \in \mathbb{S}$.

Lemma 12

$\mathbb{M}_T^{++}$ is dense in $\mathbb{M}_T$.

Lemma 13

Fix $s \in [0,T]$ and $\nu \in \mathbb{M}_T^{++}$. Then there is an optimizer $u^*(s,\cdot)$ of

$$\inf_{u \in U} \left[\sum_{i=1}^{d} \frac{(Qu)(i)}{u(i)} K_\nu(s,i) \right]$$

such that $u^*(\cdot,i) \in \mathbb{C}_{[0,T]}^\infty$, for all $i \in \mathbb{S}$.

Step functions

Let $\mathbb{S}_T$ denote the space of all step functions on $[0, T]$ of the form, for $k \in \mathbb{N}$ and real numbers $\lambda_0, \cdots, \lambda_k$,

$$\lambda(t) = \lambda_0 \mathbf{1}_{\{t=0\}}(t) + \sum_{i=0}^{k} \lambda_i \mathbf{1}_{(t_i, t_{i+1}]}(t), 0 = t_0 < \cdots < t_{k+1} = T.$$

For any $\varphi \in \mathbb{C}_T$, we introduce the following notation

$$\int_0^T \lambda(s) \mathrm{d}\varphi_s := \sum_{i=0}^{k} \lambda_i [\varphi_{T \wedge t_{i+1}} - \varphi_{T \wedge t_i}].$$

Stochastic exponential I, first density

Put

$$N_t^\epsilon := \frac{1}{\sqrt{\epsilon}} \int_0^t \lambda(s)\sigma(X_s^\epsilon, M_s^\epsilon)\mathrm{d}B_s, \quad \lambda \in \mathbb{S}_T,$$

which has the stochastic exponential

$$\mathcal{E}(N^\epsilon)_t = \exp\left(N_t^\epsilon - \frac{1}{2}\langle N^\epsilon \rangle_t \right).$$

(Nonrandom) lower bound on the first density

Lemma 14

For every $(\varphi, \nu) \in \mathbb{C}_T \times \mathbb{M}_T$ and every $\lambda \in \mathbb{S}_T$, $\delta > 0$, there exists a positive constant $K_{\lambda,\varphi,T}$ not depending on ϵ or δ such that

$$\mathcal{E}(N^\epsilon)_T \geq \exp\Bigg(\frac{1}{\epsilon}\Big(\int_0^T \lambda(s)\mathrm{d}\varphi_s - \int_0^T \lambda(s)\hat{b}_s(\nu, \varphi_s)\mathrm{d}s - \int_0^T \frac{\lambda^2(s)}{2}\hat{\sigma}_s^2(\nu, \varphi_s)\mathrm{d}s\Big) - \frac{\delta}{\epsilon}K_{\lambda,\varphi,T}\Bigg)$$

on the set $\{\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \leq \delta\}$.

Stochastic exponential II, second density

Let $\mathbb{U}$ denote the space of functions on $[0, T] \times \mathbb{S}$ continuously differentiable in $s \in [0, T]$ and $\inf_{s\in[0,T], i\in\mathbb{S}} u(s, i) > 0$.

For any $u(\cdot, \cdot) \in \mathbb{U}$,

$$\hat{N}_t^\epsilon = u(t, X_t^\epsilon) - u(0, X_0^\epsilon) - \int_0^t \frac{\partial}{\partial s} u(s, X_s^\epsilon) \mathrm{d}s - \int_0^t (Q^\epsilon u)(s, X_s^\epsilon) \mathrm{d}s$$

is a local martingale. We define

$$\tilde{N}_t^\epsilon := \int_0^t \frac{1}{u(s-, X_{s-}^\epsilon)} \mathrm{d}\hat{N}_s^\epsilon.$$

Then

$$\mathcal{E}(\tilde{N}^\epsilon)_t = \frac{u(t, X_t^\epsilon)}{u(0, X_0^\epsilon)} \exp\left(- \int_0^t \frac{\frac{\partial}{\partial s} u(s, X_s^\epsilon) + (Q^\epsilon u)(s, X_s^\epsilon)}{u(s, X_s^\epsilon)} \mathrm{d}s \right).$$

(Nonrandom) lower bound on the second density

Lemma 15

For every $\nu \in \mathbb{M}_T$, every $u \in \mathbb{U}$ and every $\gamma, \delta > 0$, there exist positive constants C_u, C'_u, K_u and $K_{Q,u}$ not depending on ϵ or δ such that

$$\mathcal{E}(\tilde{N}^\epsilon)_T \geq K_u \exp\bigg(-(C_u\delta + \gamma + C'_u T + \frac{1}{\epsilon}(K_{Q,u}\delta + \gamma))d - \frac{1}{\epsilon}\int_0^T \sum_{i=1}^d \frac{Qu(s,i)}{u(s,i)} K_\nu(s,i)\mathrm{d}s \bigg)$$

on the set $\{\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \leq \delta\}$.

Use of the lemmas

As $\mathcal{E}(\tilde{N}^\epsilon)_T \mathcal{E}(N^\epsilon)$ is a supermartingale, $\mathbb{E}\mathcal{E}(\tilde{N}^\epsilon)_T \mathcal{E}(N^\epsilon)_T \leq 1$.

Then Lemmas 12, 14, 15 imply

$$\mathbb{P}(\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \leq \delta) \leq \text{exponential upper bound.}$$

Optimizing over $\lambda \in \mathbb{S}_T$ and other parameters lead to the upper bound in the local LDP on $\mathbb{C}_T \times \mathbb{M}_T^{++}$.

Auxiliary result IV

Lemma 16

Let $\nu^\eta, \nu \in \mathbb{M}_T$ *with kernels* K^η_ν *and* K_ν *such that* $K^\eta_\nu(\cdot, i) \to K_\nu(\cdot, i)$ *a.e. as* $\eta \to 0$ *on* $[0, T]$ *for each* $i \in \mathbb{S}$*. Then*

(i) $\tilde{I}_T(\nu^\eta) \to \tilde{I}_T(\nu)$ *as* $\eta \to 0$*;*

(ii) $I_T(\varphi, \nu^\eta) \to I_T(\varphi, \nu)$ *as* $\eta \to 0$*,* $\forall \varphi \in \mathbb{H}_T$*, if* $\inf_{i,x} \sigma^2(i, x) > 0$*.*

Upper bound in the local LDP on $\mathbb{C}_T \times \mathbb{M}_T$

Lemmas 5 and 16 and lower semicontinuity of the rate functions then lead to

Proposition 17

For every $(\varphi, \nu) \in \mathbb{C}_T \times \mathbb{M}_T$,

$$\limsup_{\delta \to 0} \limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}(\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \leq \delta) \leq -L_T(\varphi, \nu).$$

Outline

Stochastic exponential III

Observe:

The rate function $I_T(\varphi, \nu)$ is finite for every $(\varphi, \nu) \in \mathbb{H}_T \times \mathbb{M}_T$ if $\inf_{i,x} \sigma^2(i,x) > 0$ (*Temporary assumption*).

Let $(\varphi, \nu) \in \mathbb{H}_T \times \mathbb{M}_T$ and put

$$\bar{N}_t^\epsilon := \frac{1}{\sqrt{\epsilon}} \int_0^t \frac{\varphi_s' - \hat{b}_s(\varphi_s, \nu)}{\hat{\sigma}(\varphi_s, \nu)} \mathrm{d}B_s =: \int_0^t h(s) \mathrm{d}B_s.$$

with stochastic exponential

$$\mathcal{E}(\bar{N}^\epsilon)_t = \exp\left(\bar{N}_t^\epsilon - \frac{1}{2} \langle \bar{N}^\epsilon \rangle_t \right).$$

Martingale property

Let, as before,

$$\mathcal{E}(\tilde{N}^\epsilon)_t = \frac{u(t, X_t^\epsilon)}{u(0, X_0^\epsilon)} \exp\left(- \int_0^t \frac{\frac{\partial}{\partial s} u(s, X_s^\epsilon) + (Q^\epsilon u)(s, X_s^\epsilon)}{u(s, X_s^\epsilon)} \mathrm{d}s \right).$$

Lemma 18

For every $(\varphi, \nu) \in \mathbb{H}_T \times \mathbb{M}_T$ and $u(\cdot,\cdot) \in \mathbb{U}$, the process $\{\mathcal{E}(\tilde{N}^\epsilon)_t \mathcal{E}(\bar{N}^\epsilon)_t\}_{t\in[0,T]}$ is a martingale if $\inf_{i,x} \sigma^2(i,x) > 0$.

Hence, with a special u^* one can define a probability measure $\mathbb{P}_{u^*} \sim \mathbb{P}$ through $\mathrm{d}\mathbb{P}_{u^*} = \mathcal{E}_T^{u^*} \mathcal{E}(\bar{N}^\epsilon)_T \mathrm{d}\mathbb{P}$, $\mathcal{E}^{u^*} = \mathcal{E}(\tilde{N}^\epsilon)$ for $u = u^*$.

(Nonrandom) lower bound on the reciprocal second density

Lemma 19

For every $\nu \in \mathbb{M}_T$, every $u \in \mathbb{U}$ and every $\gamma, \delta > 0$, there exist positive C_u, C'_u, K'_u and $K_{Q,u}$ not depending on ϵ or δ such that

$$[\mathcal{E}(\tilde{N}^\epsilon)_T]^{-1} \geq K'_u \exp\Bigg(-\,(C_u\delta + \gamma + C'_u T + \frac{1}{\epsilon}(K_{Q,u}\delta + \gamma))d + \frac{1}{\epsilon}\int_0^T \sum_{i=1}^d \frac{Qu(s,i)}{u(s,i)} K_\nu(s,i)\mathrm{d}s \Bigg)$$

on the set $\{\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \leq \delta\}$.

We will use this for the special, *optimal* u^*.

Lower bound if $\inf_{i,x} \sigma^2(i,x) > 0$

By Lemma 18 one has for $B_\delta = \{\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \le \delta\}$

$$\mathbb{P}(B_\delta) = \int_{B_\delta} \left[\mathcal{E}_T^{u^*} \mathcal{E}(\bar{N}^\epsilon)_T\right]^{-1} \mathrm{d}\mathbb{P}_{u^*}$$

and then by Lemma 19

$$\mathbb{P}(B_\delta) \ge \text{exponential lower bound} \times \mathbb{P}_{u^*}(\text{some set}),$$

which *eventually* leads to

Proposition 20

For every $(\varphi, \nu) \in \mathbb{H}_T \times \mathbb{M}_T$, if $\inf_{i,x} \sigma^2(i,x) > 0$,

$$\liminf_{\delta \to 0} \liminf_{\epsilon \to 0} \epsilon \log \mathbb{P}(\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \le \delta) \ge -L_T(\varphi, \nu).$$

Perturbed process

Next we drop the assumption $\inf_{i,x} \sigma^2(i,x) > 0$.

Given $\gamma > 0$, we consider the perturbed SDE

$$M_t^{\epsilon,\gamma} = \int_0^t b(X_s^\epsilon, M_s^{\epsilon,\gamma})\mathrm{d}s + \sqrt{\epsilon}\int_0^t \sigma(X_s^\epsilon, M_s^{\epsilon,\gamma})\mathrm{d}B_s + \sqrt{\epsilon}\gamma W_t,$$

where W_t is a Brownian motion, independent of B_t and X_t^ϵ.

$M^{\epsilon,\gamma}$ and M^ϵ are 'superexponentially close':

Lemma 21

For every $T > 0$ and $\eta > 0$,

$$\lim_{\gamma\to 0} \limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\rho_T(M^{\epsilon,\gamma}, M^\epsilon) > \eta\right) = -\infty.$$

Final lower bound for the local LDP

Combining the case $\inf_{i,x} \sigma^2(i,x) > 0$, Proposition 20 ('true' for the quadratic variation in the perturbed case), Lemma 21 and letting $\gamma \to 0$ leads to

Proposition 22

For every $(\varphi, \nu) \in \mathbb{C}_T \times \mathbb{M}$,

$$\liminf_{\delta \to 0} \liminf_{\epsilon \to 0} \epsilon \log \mathbb{P}(\rho_T(M^\epsilon, \varphi) + d_T(\nu^\epsilon, \nu) \leq \delta) \geq -L_T(\varphi, \nu).$$

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Thank you!