Aharonov-Bohm Effect in Resonances for Magnetic Scattering by Three Solenoids

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0. Aharonov–Bohm effect (AB effect)

We work in two dimensions and denote by

$$H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2} (-i\partial_j - a_j)^2 ,$$

the magnetic Schrödinger operator with the potential

$$A = (a_1, a_2) : R^2 \to R^2.$$

$$b = \nabla \times A = \partial_1 a_2 - \partial_2 a_1 : R^2 \to R \quad \text{(magnetic field)}$$

$$\alpha = (2\pi)^{-1} \int b(x)\, dx \quad \text{(magnetic flux)}$$

$$\operatorname{supp} b \subset \operatorname{supp} A, \quad \operatorname{supp} b \neq \operatorname{supp} A \text{ (in general)}.$$

The Aharonov–Bohm potential (AB potential)

$$A = (-x_2/|x|^2, x_1/|x|^2) = (-\partial_2 \log|x|, \partial_1 \log|x|)\,, \quad \operatorname{supp} A = R^2,$$

defines the solenoid (δ–like magnetic field)

$$b = \nabla \times A = (\partial_1^2 + \partial_2^2)\log|x| = 2\pi\delta(x), \quad \operatorname{supp} b = \{0\}.$$

The Aharonov–Bohm effect (the AB effect) says that a vector potential influences quantum particles, even if they move over a region where the corresponding magnetic field vanishes.

1. Problem

We consider the scattering system by three solenoids. Assume that we are given the three centers

$$d_- = (-\kappa_- d, 0), \quad d_0 = (0, \kappa\, d^{1/2}), \quad d_+ = (\kappa_+ d, 0),$$

where $\kappa_\pm > 0$ with $\kappa_- + \kappa_+ = 1$ and

$$|d_+ - d_-| = d \gg 1 \text{ (regarded as a large parameter)}.$$

We again denotes by $A(x)$ the AB potential and define

$$A_d(x) = \alpha_- A(x - d_-) + \alpha_0 A(x - d_0) + \alpha_+ A(x - d_+),$$

where real numbers $\alpha_\pm$, α_0 denote magnetic fluxes.

$$\nabla \times A_d = 2\pi \left(\alpha_- \delta(x - d_-) + \alpha_0 \delta(x - d_0) + \alpha_+ \delta(x - d_+)\right).$$

We consider magnetic Schrödinger operator

$$H_d = H(A_d) \quad \text{in } L^2 = L^2(R^2).$$

It acts as a symmetric operator on $C_0^\infty(R^2 \setminus \{d_-, d_0, d_+\})$, but it is not necessarily essentially self–adjoint. The Friedrichs extension H_d is defined under the boundary conditions

$$\lim_{|x-c|\to 0} |u(x)| < \infty, \quad c = d_-,\ d_0,\ d_+.$$

We define the resonance of H_d. The resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2 \to L^2, \quad \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta > 0,$$

is bounded, and it admits the meromorphic extension over the lower half plane ($\operatorname{Im}\zeta \le 0$) as a function with values in operators

$$R(\zeta; H_d) : L^2_{\text{comp}} \to L^2_{\text{loc}},$$

$$L^2_{\text{comp}} = \{u \in L^2 : \operatorname{supp} u \text{ is compact}\}.$$

The resonance is defined as the pole of the meromorphic function. For $d = |d_+ - d_-| \gg 1$, the resonances are created near the positive axis by the trajectories trapped between the two centers d_- and d_+.

We study

· how the potential $\alpha_0 A(x - d_0)$ associated with the third solenoid $2\pi\delta(x - d_0)$ influences the location of resonances by the AB effect.

We also discuss

· what happens in the case of four solenoids.

2. Heuristic arguments

The scattering system by one solenoid $2\pi\alpha\delta(x)$ is known as a solvable model in quantum mechanics. We denote by

$$f(\omega \to \theta; E), \quad \omega,\ \theta \in S^1, \quad E > 0,$$

the amplitude for scattering from ω to θ at energy $E > 0$.
The backward amplitude takes the form (independent of ω)

$$f(\omega \to -\omega; E) = (2\pi)^{-1/2} e^{i\pi/4}(-1)^{[\alpha]+1} \sin(\alpha\pi) E^{-1/4},$$

where $[\alpha]$ denotes the Gauss notation (the greatest integer not exceeding α). We note that $f(\omega \to -\omega; E) = 0$ for integer flux α.

We write

$$f_{\pm}(\omega \to -\omega; E)$$

for the backward amplitude by the solenoid $2\pi\alpha_{\pm}\delta(x)$.

We denote by

$$\varphi_0(x;\omega,E) = \exp(iE^{1/2}x\cdot\omega), \quad \omega \in S^1, \quad E > 0.$$

the plane wave with incident direction ω at energy E.

We use the notation: $\omega_1 = (1,0)$ and $x_\pm = x - d_\pm$.

We study the trapping phenomenon between d_- and d_+.

We consider the special case $\kappa = 0$ ($d_0 = (0,0)$).

We take d_- as the origin and consider the wave $\varphi_0(x_-; -\omega_1, E)$.

It hits $2\pi\alpha_-\delta(x_-)$ and is scattered into direction ω_1.

The wave is scattered as the spherical wave

$$f_-(-\omega_1 \to \omega_1; E) \exp\left(iE^{1/2}|x_-|\right) |x_-|^{-1/2} \times (\text{AB effect term}).$$

We take d_+ as the origin and calculate

$$\begin{aligned} |x_-| = |x - d_-| &= |d_+ - d_- + x - d_+| \\ &= |d\,\omega_1 + x_+| \sim d + \omega_1 \cdot x_+, \quad d \gg 1, \end{aligned}$$

around d_+. The spherical wave behaves like the plane wave

$$\exp\left(iE^{1/2}|x_-|\right) |x_-|^{-1/2} \sim \left(e^{iE^{1/2}d}/d^{1/2}\right) \varphi_0(x_+; \omega_1, E)$$

around d_+.
The scattered wave hits the other solenoid $2\pi\alpha_+\delta(x_+)$ as the plane wave

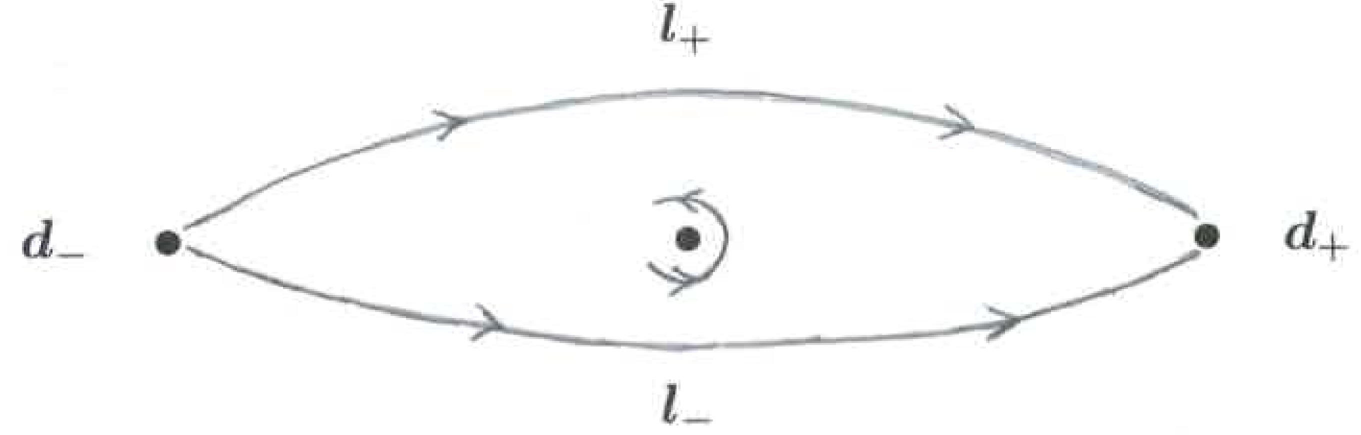
$$\left(e^{iE^{1/2}d}/d^{1/2}\right) f_-(-\omega_1 \to \omega_1; E) \times (\text{AB effect term}) \times \varphi_0(x_+; \omega_1, E).$$

Recall that $d_0 = (0, 0)$. Consider the AB effect from the potential

$$A_0(x) = \alpha_0 A(x), \quad \nabla \times A_0 = 2\pi\alpha_0\delta(x).$$

The wave function changes the phase factor given by the line integral

$$\int_{l_\pm} A_0(x) \cdot dx = \mp\alpha_0\pi \quad \text{(Stokes formula)}.$$



$$0.5 \times \exp(-i\alpha_0\pi) + 0.5 \times \exp(i\alpha_0\pi) = \cos(\alpha_0\pi).$$

The wave scattered by $2\pi\alpha_-\delta(x_-)$ behaves like the plane wave

$$\left(e^{iE^{1/2}|d|}/|d|^{1/2}\right) f_-(-\omega_1 \to \omega_1; E)\cos(\alpha_0\pi)\varphi_0(x_+;\omega_1,E)$$

around d_+ and it hits the other solenoid $2\pi\alpha_+\delta(x_+)$.

The same argument applies to $\varphi_0(x_+;\omega_1,E)$. The wave scattered by $2\pi\alpha_+\delta(x_+)$ takes a similar form

$$\left(e^{iE^{1/2}d}/d^{1/2}\right) f_+(\omega_1 \to -\omega_1; E)\cos(\alpha_0\pi)\varphi_0(x_-;-\omega_1,E)$$

around d_-.

The first plane wave $\varphi_0(x_-;-\omega_1,E)$ returns to d_-. Then it takes the form $g(E;d)\varphi_0(x_-;-\omega_1,E)$, where

$$g = \left(e^{2iE^{1/2}d}/d\right) f_-(-\omega_1 \to \omega_1; E) f_+(\omega_1 \to -\omega_1; E)\cos^2(\alpha_0\pi).$$

The trapping phenomenon is described by the series

$$\left(\sum_{n=0}^{\infty} g(E;d)^n\right)\varphi_0(x_-;-\omega_1,E)\,(<\infty)\,,\quad E>0.$$

The scattering amplitude

$$\zeta \mapsto f_{\pm}(\pm\omega_1 \to \mp\omega_1; \zeta) \sim (\mathrm{const}) \times \zeta^{-1/4}$$

admits the analytic extension over the lower half plane $\mathrm{Im}\,\zeta \leq 0$. The resonances are approximately specified as solutions to equation

$$g(\zeta; d) = \left(\frac{e^{2ikd}}{d}\right) f_{-}(-\omega_1 \to \omega_1; \zeta) f_{+}(\omega_1 \to -\omega_1; \zeta) \cos^2(\alpha_0 \pi) = 1,$$

where $k = \zeta^{1/2}$ ($\mathrm{Re}\,k > 0$ for $\mathrm{Re}\,\zeta > 0$).

$$|\exp(2ikd)| = \exp(-2d\,(\mathrm{Im}\,k)) \gg 1, \quad \mathrm{Im}\,\zeta < 0.$$

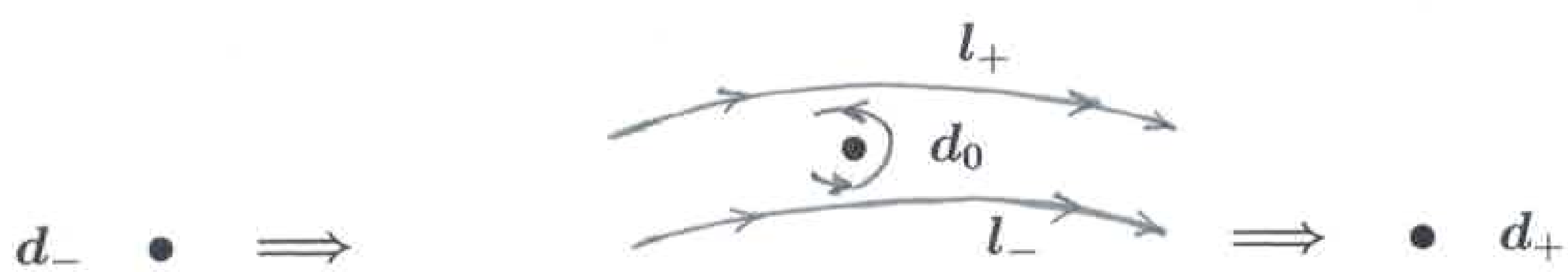
The relation makes sense only when $\cos(\alpha_0\pi) \neq 0$ and

$$f_{\pm}(\pm\omega_1 \to \mp\omega_1; \zeta) \neq 0 \quad (\Leftrightarrow \alpha_{\pm} \text{ is not an integer}).$$

3. Formulation of result (three solenoids)

We discuss the general case $\kappa \neq 0$ ($d_0 = (0, \kappa\, d^{1/2})$).
We consider the AB effect for the trajectories from $d_- = (-\kappa_- d, 0)$ to $d_+ = (\kappa_+ d, 0)$. We define the Fresnel–type integral

$$I(\zeta) = (2/\pi)^{1/2}\, e^{-i\pi/4} \int_0^{\tau} e^{is^2/2}\, ds, \;\; \tau(\zeta) = \kappa\,(1/\kappa_- + 1/\kappa_+)^{1/2}\, \zeta^{1/4}.$$



The AB effect term is determined by

$$\pi_-(\zeta) = \{(1 + I(\zeta))\,/2\}\exp(i\alpha_0\pi) + \{(1 - I(\zeta))\,/2\}\exp(-i\alpha_0\pi)\,.$$

The AB effect term for the trajectories from d_+ to d_- is given by

$$\pi_+(\zeta) = \{(1 + I(\zeta))\,/2\}\exp(-i\alpha_0\pi) + \{(1 - I(\zeta))\,/2\}\exp(i\alpha_0\pi)\,.$$

We fix $E_0 > 0$ and take a neighborhood

$$D_d = \left\{\zeta : |\mathrm{Re}\,\zeta - E_0| < \delta E_0,\ |\mathrm{Im}\,\zeta| < (1 + 2\delta)\, E_0^{1/2}\,((\log\, d)/d)\right\}$$

for $0 < \delta \ll 1$ small enough. We consider

$$g(\zeta; d) = \left(e^{2ikd}/d\right) f_0(\zeta)\pi_-(\zeta)\pi_+(\zeta), \quad k = \zeta^{1/2}, \quad \zeta \in D_d,$$

$$\begin{aligned} f_0(\zeta) &= f_-(-\omega_1 \to \omega_1; \zeta) f_+(\omega_1 \to -\omega_1; \zeta) \\ &= (2\pi)^{-1}\, i(-1)^{[\alpha_-]+[\alpha_+]} \sin(\alpha_-\pi) \sin(\alpha_+\pi)\zeta^{-1/2}. \end{aligned}$$

We can take $\delta > 0$ so small that

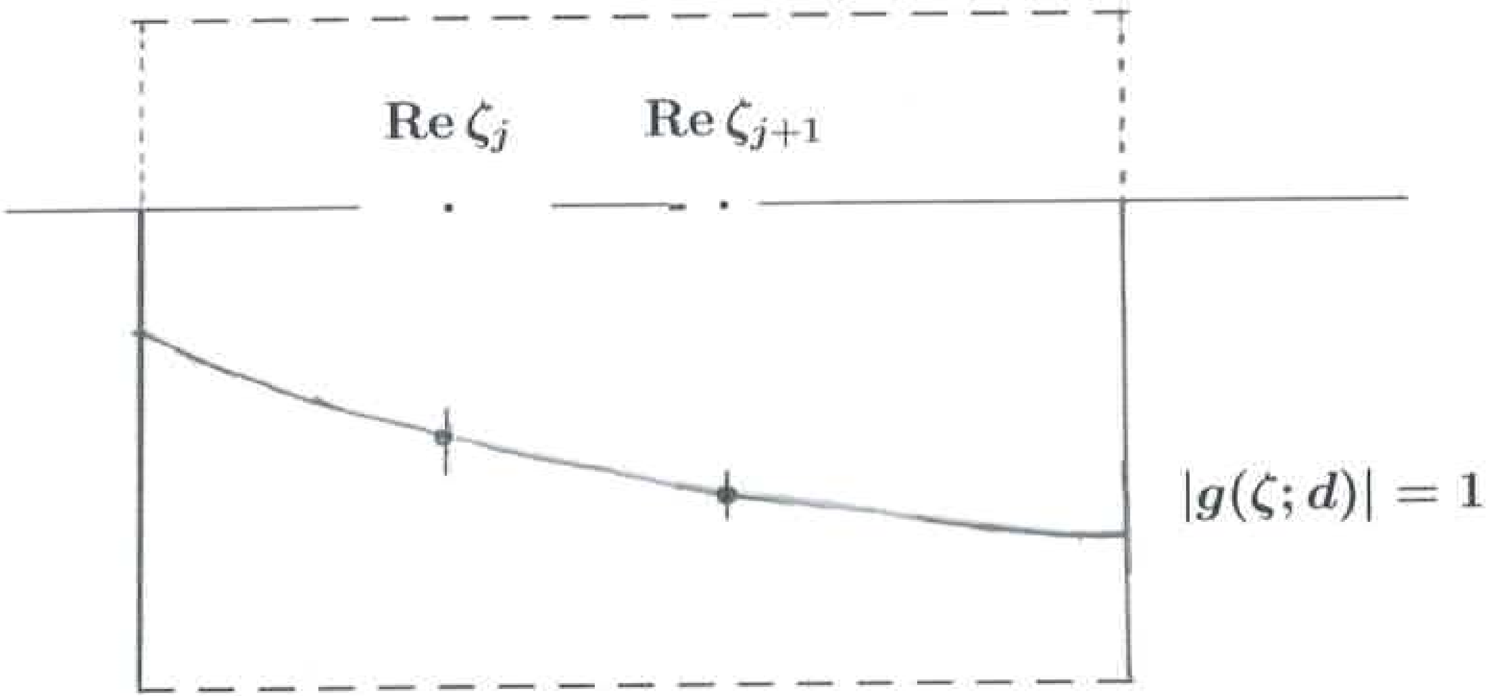
$$d^{\delta} < |\exp(2ikd)|\,/d < d^{3\delta}, \quad d \gg 1,$$

on the bottom of D_d ($\mathrm{Im}\,\zeta = -\,(1 + 2\delta)\, E_0^{1/2}\,((\log|d|)\,/|d|)$).
This implies that $\{\ |g(\zeta; d)| = 1 : |\mathrm{Re}\,\zeta - E_0| < \delta E_0\} \subset D_d$

We consider the equation $g(\zeta; d) = 1$ in D_d. The solutions

$$\{\zeta_j(d)\}, \quad \mathrm{Re}\,\zeta_1 < \mathrm{Re}\,\zeta_2 < \cdots < \mathrm{Re}\,\zeta_{N_d},$$

are distributed as follows:



$$\mathrm{Im}\,\zeta_j \sim -E_0^{1/2}\,(\log\, d)\,/d, \qquad \mathrm{Re}\,(\zeta_{j+1} - \zeta_j) \sim E_0^{1/2}(2\pi/d).$$

Theorem 1 Assume that $\pi_{\pm}(E_0) \neq 0$ at energy $E_0 > 0$ and that $\alpha_{\pm}$ is not an integer. Then we can take $\delta > 0$ in the following way: For any $\varepsilon > 0$ small enough, there exists $d_\varepsilon \gg 1$ such that for $d > d_\varepsilon$, H_d has the resonances

$$\{\zeta_{\mathrm{res},j}(d)\}, \quad \mathrm{Re}\,\zeta_{\mathrm{res},1}(d) < \mathrm{Re}\,\zeta_{\mathrm{res},2}(d) < \cdots < \mathrm{Re}\,\zeta_{\mathrm{res},N_d}(d)$$

in the neighborhood $\{\zeta \in D_d : |\zeta - \zeta_j(d)| < \varepsilon/d\}$.
Moreover, the resolvent

$$R(\zeta; H_d) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$$

is analytic over $D_d \setminus \{\zeta_{\mathrm{res},1}(d), \zeta_{\mathrm{res},2}(d), \cdots, \zeta_{\mathrm{res},N_d}(d)\}$.

• If $|\kappa| \gg 1$, then the AB effect is not observed (loosely speaking). Let $\kappa \gg 1$. The contribution from trajectory l_+ is neglected.

$$\tau(\zeta) = \kappa\,(1/\kappa_- + 1/\kappa_+)^{1/2}\,\zeta^{1/4} \to \infty,$$

$$I(\zeta) \sim (2/\pi)^{1/2}\,e^{-i\pi/4} \int_0^\infty e^{is^2/2}\,ds = 1.$$

$$\pi_-(\zeta) \sim ((1 + I(\zeta))\,/2)\exp(i\alpha_0\pi) \sim \exp(i\alpha_0\pi),$$

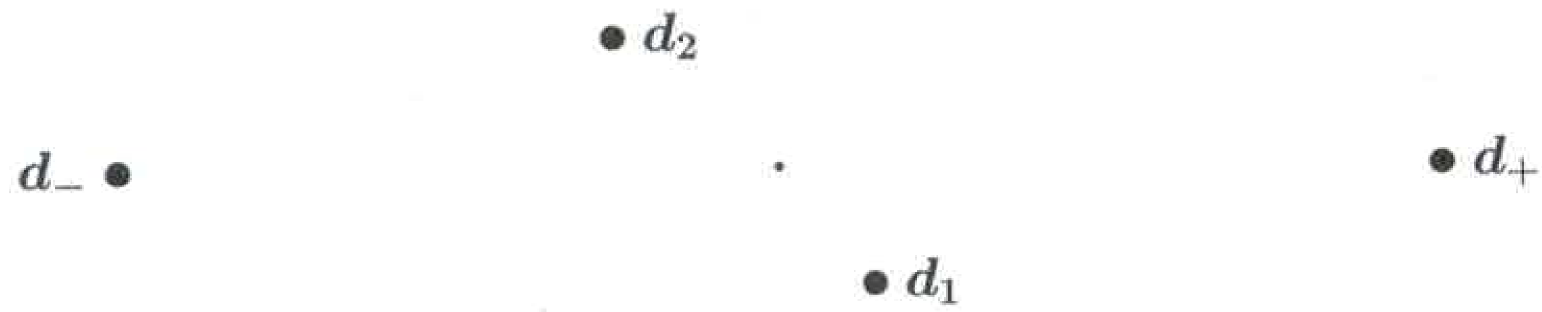
$$\pi_+(\zeta) \sim \exp(-i\alpha_0\pi), \quad \pi_-(\zeta)\pi_+(\zeta) \sim 1.$$

4. Four solenoids

Assume that we are given the four centers

$$d_\mp = (\mp\kappa_\mp d, 0), \quad d_1 = (-\kappa_0 d, \kappa_1 d^{1/2}), \quad d_2 = (-\kappa_0 d, \kappa_2 d^{1/2}),$$

where $0 \le \kappa_0 < \min\,(\kappa_-, \kappa_+)$.



We use the same notation. We set

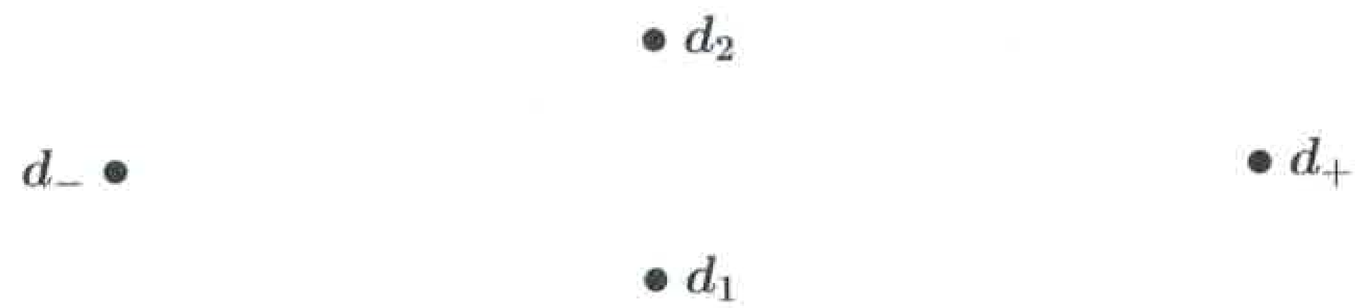
$$A_d(x) = \alpha_- A(x - d_-) + \alpha_1 A(x - d_1) + \alpha_2 A(x - d_2) + \alpha_+ A(x - d_+)$$

and consider the self-adjoint operator $H_d = H(A_d)$ under the boundary conditions

$$\lim_{|x-c|\to 0} |u(x)| < \infty, \quad c = d_-,\ d_1,\ d_2,\ d_+.$$

No results in the general case. We discuss the two special cases.
(1) horizontal case ($\kappa_1 = \kappa_2 = 0$) : $d_1 = (-\kappa_0 d, 0)$, $d_2 = (\kappa_0 d, 0)$

(2) vertical case ($\kappa_0 = 0$) : $d_1 = (0, \kappa_1 d^{1/2})$, $d_2 = (0, \kappa_2 d^{1/2})$

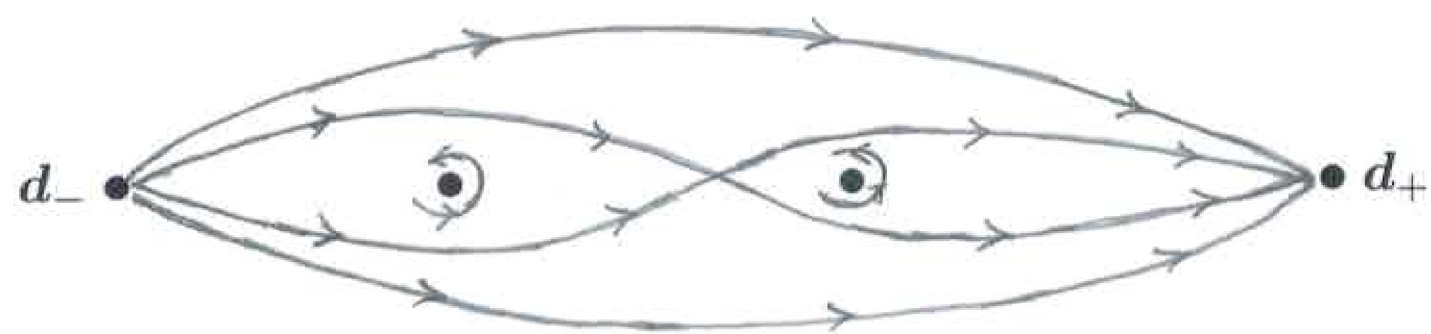


4.1. Horizontal case

We set

$$\beta_+ = \alpha_2 + \alpha_1, \qquad \beta_- = \alpha_2 - \alpha_1.$$

There are four kinds of trajectories from d_- to d_+.



The phase factor changes along these trajectories:

$$(\cdots)\cos(\beta_+\pi) + (\cdots)\cos(\beta_-\pi).$$

The AB effect term depends on the distances between centers.

We define the angle ω_0 by

$$\cos\omega_0 = \left(\frac{\kappa_- - \kappa_0}{\kappa_- + \kappa_0}\right)^{1/2}\left(\frac{\kappa_+ - \kappa_0}{\kappa_+ + \kappa_0}\right)^{1/2} < 1, \quad 0 < \omega_0 < \pi/2.$$

The AB effect term is given by

$$\pi_0 = (1 - \omega_0/\pi)\cos(\beta_+\pi) + (\omega_0/\pi)\cos(\beta_-\pi)$$

for the trajectories from d_- to d_+.
A similar relation remains true for the trajectories from d_+ to d_-.

We define

$$g_1(\zeta; d) = \left(e^{2ikd}/d\right) f_0(\zeta)\pi_0^2$$

over D_d, where $f_0(\zeta)$ is again defined by

$$f_0(\zeta) = f_-(-\omega_1 \to \omega_1; \zeta) f_+(\omega_1 \to -\omega_1; \zeta).$$

Theorem 2 Assume that $\pi_0 \neq 0$ and $\alpha_\pm$ is not an integer. Then the resonances are approximately determined by

$$g_1(\zeta; d) = 1, \qquad \zeta \in D_d,$$

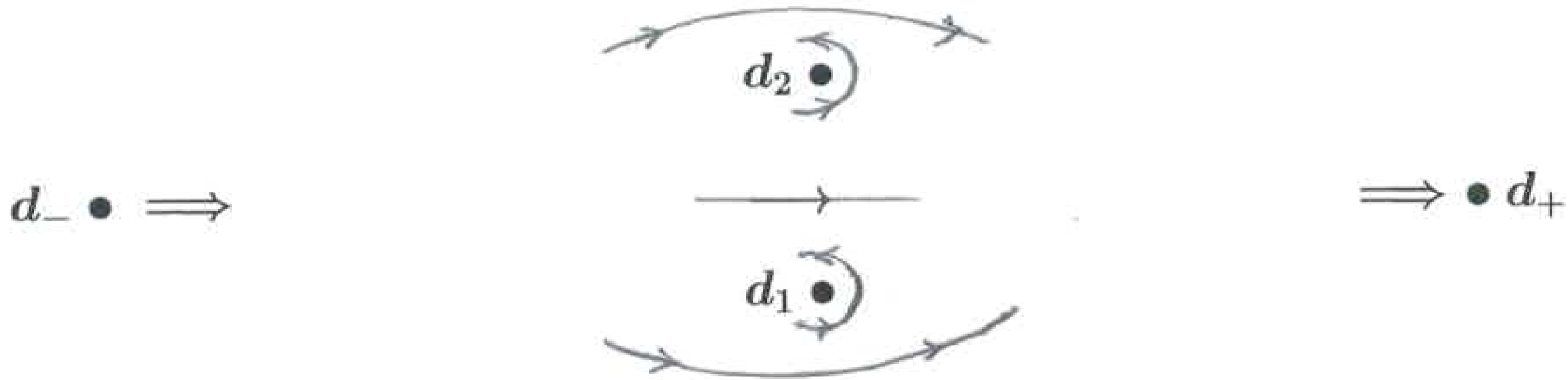
as in Theorem 1.

4.2. Vertical case

Recall the notation :

$$d_j = (0, \kappa_j d^{1/2})\ (j = 1, 2), \quad \beta_+ = \alpha_2 + \alpha_1, \quad \beta_- = \alpha_2 - \alpha_1.$$

Assume that $\kappa_2 > \kappa_1$ (without loss of generality).
There are three kinds of trajectories from d_- to d_+.



The phase factor changes along these trajectories:

$$(\cdots) \exp(i\beta_+\pi) + (\cdots) \exp(i\beta_-\pi) + (\cdots) \exp(-i\beta_+\pi).$$

The AB effect term depends on the Fresnel–type integrals.
We define

$$I_j(\zeta) = (2/\pi)^{1/2} e^{-i\pi/4} \int_0^{\tau_j} e^{is^2/2} ds,$$
$$\tau_j(\zeta) = \kappa_j (1/\kappa_- + 1/\kappa_+)^{1/2} \zeta^{1/4},$$

for $j = 1, 2$.
The AB effect term is given by

$$\rho_-(\zeta) = p_1(\zeta) \exp(i\beta_+\pi) + p(\zeta) \exp(i\beta_-\pi) + p_2(\zeta) \exp(-i\beta_+\pi)$$

for the trajectories from d_- to d_+, where

$$p_1 = (1 + I_1(\zeta))/2, \quad p_2 = (1 - I_2(\zeta))/2, \quad p = (I_2(\zeta) - I_1(\zeta))/2.$$

For the trajectories from d_+ to d_-, the AB effect term is given by

$$\rho_+(\zeta) = p_1(\zeta) \exp(-i\beta_+\pi) + p(\zeta) \exp(-i\beta_-\pi) + p_2(\zeta) \exp(i\beta_+\pi).$$

We define

$$g_2(\zeta; d) = \left(e^{2ikd}/d\right) f_0(\zeta)\rho_-(\zeta)\rho_+(\zeta)$$

over D_d.

Theorem 3 Assume that $\rho_\pm(E_0) \neq 0$ and $\alpha_\pm$ is not an integer. Then the resonances are approximately determined by

$$g_2(\zeta; d) = 1, \qquad \zeta \in D_d,$$

as in Theorem 1.

5. Strategy of proof of Theorem 1 (three solenoids)

5.1. Resolvent kernel for one solenoid

The scattering system by one solenoid is solvable.

We consider the self–adjoint operator

$$H = H(A_0), \quad A_0 = \alpha_0 A(x), \quad \nabla \times A_0 = 2\pi \alpha_0 \delta(x),$$

under the boundary condition $\lim_{|x| \to 0} |u(x)| < \infty$ at the origin.
The operator H is expanded as the partial waves

$$H \simeq \sum_{l \in Z} \oplus \left(-\partial_r^2 + (\nu^2 - 1/4) r^{-2}\right) \otimes Id, \quad \nu = |l - \alpha_0|,$$

on $L^2(R^2) \simeq L^2(0, \infty) \otimes L^2(S^1)$.

We write

$$x = (|x| \cos\theta, |x| \sin\theta), \quad y = (|y| \cos\omega, |y| \sin\omega),$$

in the polar coordinates.
The resolvent kernel $R(\zeta; H)(x, y)$ is expanded as

$$R(\zeta; H)(x, y) = (i/4) \sum_l e^{il(\theta-\omega)} J_\nu (k(|x| \wedge |y|))\, H_\nu (k(|x| \vee |y|)),$$

where $H_\nu = H_\nu^{(1)}$ is the Hankel function of first kind and

$$k = \zeta^{1/2}, \quad |x| \wedge |y| = \min(|x|, |y|), \quad |x| \vee |y| = \max(|x|, |y|).$$

The kernel grows exponentially for ζ with $\operatorname{Im} \zeta < 0$.

- One solenoid system has no resonances in $C \setminus \{0\}$.

The resolvent kernel admits the decomposition

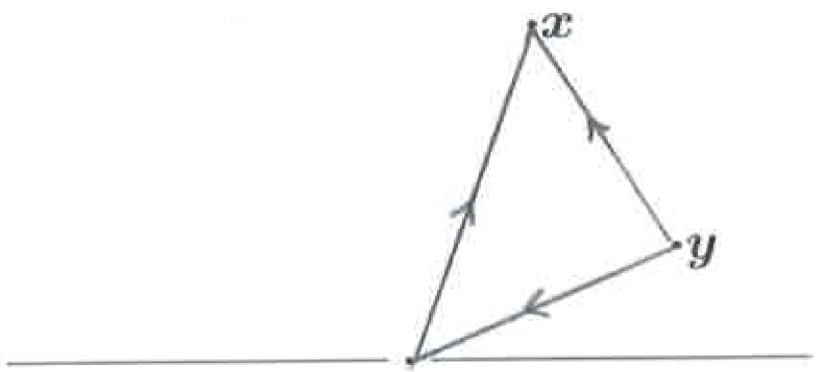
$$R(\zeta; H)(x, y) = G_{\mathrm{fr}}(x, y; \zeta) + G_{\mathrm{sc}}(x, y; \zeta).$$

- $G_{\mathrm{fr}}(x, y; \zeta)$ corresponds to the free trajectory.

  A particle goes to x from y directly without being scattered by the solenoid at the origin.

- $G_{\mathrm{sc}}(x, y; \zeta)$ corresponds to the scattering trajectory.

  A particle goes to x from y after scattered by the solenoid at the origin.



$G_{\text{fr}}(x, y; \zeta)$ behaves like

$$G_{\text{fr}} \sim e^{i\alpha\psi} H_0(k|x-y|), \quad |x-y| \gg 1,$$

where $\psi = \theta - \omega$ ($|\psi| < \pi$). We skip some numerical constants.
We write $\hat{x} = x/|x|$ for the direction of x.
G_{fr} is singular along forward directions

$$\hat{x} = -\hat{y} \quad (\psi = \theta - \omega = \pm\pi) .$$

$G_{\text{fr}}(x, y; \zeta) \sim e^{\pm i\alpha\pi} H_0(k|x-y|)$ is not continuous.

$G_{\rm sc}(x, y; \zeta)$ behaves like

$$G_{\rm sc} \sim f(-\hat{y} \to \hat{x}; \zeta)\overline{(\exp(-i\bar{k}|y|)/|y|^{1/2})}\left(\exp(ik|x|)/|x|^{1/2}\right),$$

where

$$f(-\hat{y} \to \hat{x}; \zeta) \sim \sin(\alpha_0\pi)\left(\frac{e^{i([\alpha_0]+1)(\theta+\omega)}}{1 - e^{i(\theta+\omega)}}\right)\zeta^{-1/4}$$

denotes the amplitude for scattering from $-\hat{y}$ to $\hat{x}$ by the solenoid $2\pi\alpha_0\delta(x)$. The denominator

$$1 - e^{i(\theta+\omega)} = 0, \qquad \hat{x} \sim \theta = -\omega \sim -\hat{y},$$

along the forward direction.
The forward amplitude $f(-\hat{y} \to -\hat{y}; \zeta)$ is divergent.

$G_{\rm sc}(x, y; \zeta)$ is also singular along forward directions.

The two singularities are canceled and

$$R(\zeta; H)(x, y) \sim \cos(\alpha_0\pi) H_0(k|x - y|), \quad |x - y| \gg 1,$$

along forward directions. In particular,

$$R(\zeta; H)(d_\pm, d_\mp) \sim \cos(\alpha_0\pi) H_0(k\, d).$$

In the general case when $H = H(\alpha_0 A(\cdot - d_0))$ $(\kappa \neq 0)$,

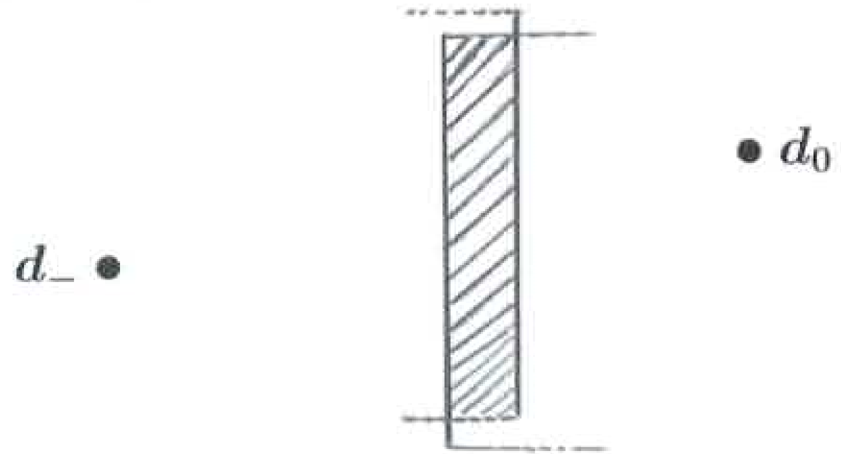
$$R(\zeta; H)(d_\pm, d_\mp) \sim \pi_\mp(\zeta) H_0(k\, d),$$

where $\pi_\mp(\zeta)$ is the AB effect term for the trajectory from $d_\mp$ to $d_\pm$.

This relation plays a basic role in proving Theorem 1.

5.2. Resolvent kernel for two solenoids

By gauge transformations, we separate the two centers from each other, and we compose the two resolvent kernels for each center.



We control the integral over the intersection by a complex scaling

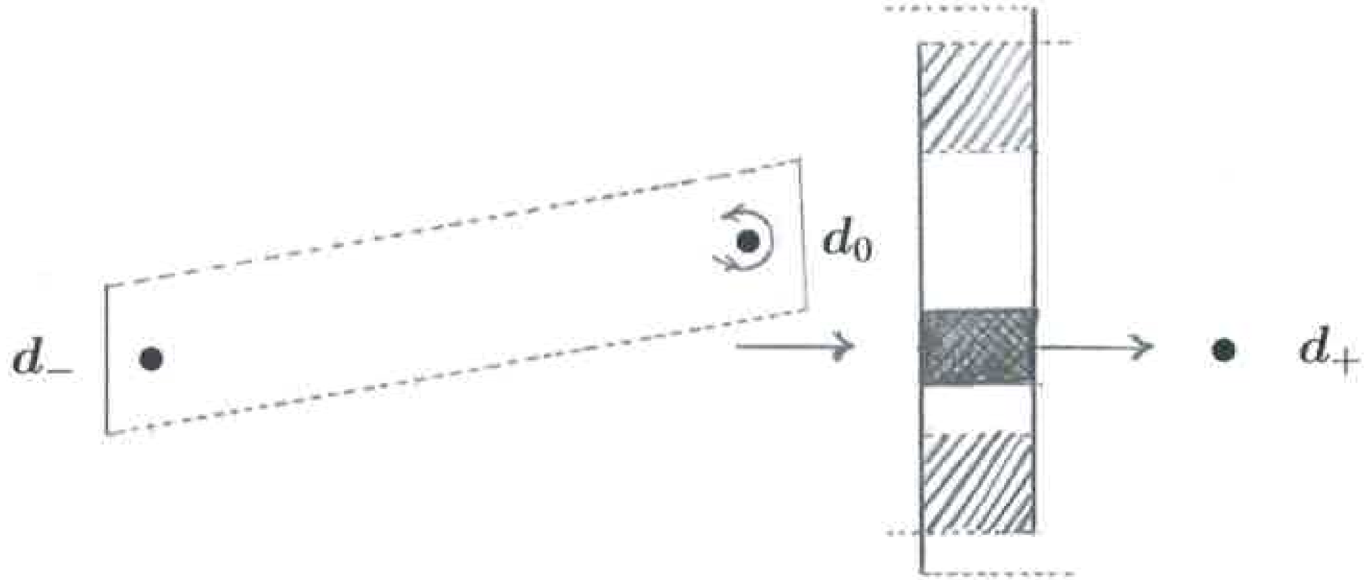
$$x_2 \to (\text{some complex variable}), \quad |x_2| \gg d.$$

This makes $\mathrm{Im}\,(k \times (\text{new variable})) > 0$ even for ζ with $\mathrm{Im}\,\zeta < 0$, and the composition of two kernels is convergent.

- $|d_0 - d_-| \sim \kappa_- d < d \Longrightarrow$ no resonances in D_d.

5.3. Resolvent kernel for three solenoids

By composition, the resolvent kernel for three solenoids is constructed from resolvent kernels for $\{d_-, d_0\}$ and for d_+.



Use a complex scaling for $|x_2| \gg d$. For trapping region $|x_2| \preceq d^{1/2}$, the asymptotic analysis along the forward direction is used for

$$R(\zeta; H)(x, d_-), \quad H = H(A_0), \quad A_0(x) = \alpha_0 A(x - d_0).$$

Appendix: semi–classical version

The semi–classical parameter h is regarded as $1/d$.

Consider the operator $(-ih\nabla - B)^2$, where

$$B(x) = \gamma_- A(x - p_-) + \gamma_0 A(x - p_{0h}) + \gamma_+ A(x - p_+),$$

$$p_- = (-\kappa_-, 0), \quad p_{0h} = (0, \kappa\, h^{1/2}), \quad p_+ = (\kappa_+, 0), \quad \kappa_\pm > 0.$$

Then

$$(-ih\nabla - B)^2 \sim H(A_d) \quad \text{(unitarily transformed)},$$

$$A_d(x) = \alpha_- A(x - d_-) + \alpha_0 A(x - d_0) + \alpha_+ A(x - d_+),$$

$$d_- = (-\kappa_- d, 0), \quad d_0 = (0, \kappa\, d^{1/2}), \quad d_+ = (\kappa_+ d, 0), \quad d = 1/h,$$

$$\alpha_\pm = (\gamma_\pm/h) - [\gamma_\pm/h], \quad \alpha_0 = (\gamma_0/h) - [\gamma_0/h].$$