Effective models in discrete magnetic Bloch systems

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Joint ongoing work with B. Helffer, I. Herbst, V. Iftimie, G. Nenciu and R. Purice

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The plan of this talk

The setting + spectral stability with respect to bounded magnetic field perturbations.

Problem 1: construction of a magnetic matrix unitary equivalent with the band Hamiltonian and its rewriting as a 'Peierls substituted', Weyl quantized ΨDO.

Problem 2: prove that given a magnetic field perturbation of strength ϵ, the spectrum moves at most like $\epsilon^{1/2}$.

Problem 3: prove that given a slowly varying magnetic field perturbation of strength ϵ, the spectral edges move like ϵ.

Problem 4: when does a slowly varying magnetic field perturbation of strength ϵ create gaps of order ϵ?

The unperturbed operator

V is a bounded, $\mathbb{Z}^d$-periodic scalar potential with $d = 2$ or $d = 3$, $H_0 = -\Delta + V$ and σ_0 is an isolated spectral island of H_0 which consists of the range of $N \geq 1$ Bloch bands.

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We know [Helffer and Sjöstrand 1989, Nenciu 1991, Panati 2007, H.C., Herbst and Nenciu 2014, Panati and Monaco 2014] that if $d \leq 3$ we can construct N exponentially localized composite Wannier functions $\{w_j\}_{j=1}^N$:

$$P_0 = \sum_{j=1}^{N} \sum_{\gamma \in \mathbb{Z}^2} |\tau_\gamma^0(w_j)\rangle\langle\tau_\gamma^0(w_j)|, \quad P_0(\mathbf{x}, \mathbf{x}') = \sum_{j=1}^{N} \sum_{\gamma \in \mathbb{Z}^2} w_j(\mathbf{x} - \gamma)\overline{w_j(\mathbf{x}' - \gamma)}.$$

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Define $\mathbf{a}(\mathbf{x}) := \int_0^1 s\mathbf{B}(s\mathbf{x}) \wedge \mathbf{x}\, ds$. Here we assume that

$$\max_{j \in \{1,2,3\}} ||B_j||_{C^1(\mathbb{R}^d)} \leq 1.$$

The magnetic phase

Denote the magnetic flux of a unit magnetic field through a triangle with corners at 0, $\mathbf{x}$ and $\mathbf{x}'$ by:

$$\phi(\mathbf{x}, \mathbf{x}') = \int_0^1 \mathbf{a}(\mathbf{x}' + s(\mathbf{x} - \mathbf{x}')) \cdot (\mathbf{x} - \mathbf{x}') ds = -\phi(\mathbf{x}', \mathbf{x}).$$

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If $d = 2$ and $\mathbf{B} = [0, 0, 1]$:

$$\phi(\mathbf{x}, \mathbf{x}') = -\frac{1}{2}\mathbf{B} \cdot (\mathbf{x} \wedge \mathbf{x}') = \frac{1}{2}(x_2 x_1' - x_2' x_1).$$

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An important estimate is the following:

$$fl(\mathbf{x}, \mathbf{y}, \mathbf{x}') := \phi(\mathbf{x}, \mathbf{y}) + \phi(\mathbf{y}, \mathbf{x}') - \phi(\mathbf{x}, \mathbf{x}'), \quad |fl(\mathbf{x}, \mathbf{y}, \mathbf{x}')| \leq |\mathbf{x} - \mathbf{y}|\ |\mathbf{y} - \mathbf{x}'|.$$

Spectral stability

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Theorem

Fix a compact set $K \subset \rho(H_0)$. Then there exist $b_0 > 0$, $\alpha < \infty$ and $C < \infty$ such that for every $0 \leq b \leq b_0$ we have that $K \subset \rho(H_b)$ and:

$$\sup_{z \in K} \left| (H_b - z)^{-1}(\mathbf{x}, \mathbf{x}') - e^{ib\phi(\mathbf{x},\mathbf{x}')}(H_0 - z)^{-1}(\mathbf{x}, \mathbf{x}') \right| \leq C\, b\, e^{-\alpha|\mathbf{x}-\mathbf{x}'|}.$$

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Thus H_b has an isolated spectral island σ_b close to σ_0. Applying the Riesz integral formula we obtain:

$$\left| P_b(\mathbf{x},\mathbf{x}') - e^{ib\phi(\mathbf{x},\mathbf{x}')} P_0(\mathbf{x},\mathbf{x}') \right| \le C\, b\, e^{-\alpha|\mathbf{x}-\mathbf{x}'|}.$$

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$$|\Xi_{\gamma,j,b}(\mathbf{x})| \leq Ce^{-\alpha|\mathbf{x}-\gamma|} \quad \text{and} \quad \mathrm{Ran}(P_b) = \overline{\mathrm{Span}\{\Xi_{\gamma,j,b}\}},$$

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2. If the field is constant, show that:

$$P_b = \sum_{\gamma \in \mathbb{Z}^d} \sum_{j=1}^{N} |\tau_\gamma^b(w_{j,b})\rangle\langle\tau_\gamma^b(w_{j,b})|, \quad [\tau_\gamma^b(f)](\mathbf{x}) = e^{ib\phi(\mathbf{x},\gamma)} f(\mathbf{x}-\gamma).$$

Important consequences

1. The restriction of H_b to the range of P_b is unitarily equivalent with a bounded operator $T_b : l^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \mapsto l^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ given by the matrix elements:

$$T_b(\gamma, j; \gamma', j') = \langle \Xi_{\gamma,j,b} | H_b \, \Xi_{\gamma',j',b} \rangle.$$

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3. If the field is constant, $\widetilde{T_b}(\gamma, j; \gamma', j') := e^{-ib\phi(\gamma,\gamma')} T_b(\gamma, j; \gamma', j')$ depends on $\gamma - \gamma'$ and it can be diagonalized by a Floquet unitary.

4. Let the field be constant. Let $\Omega = [-1/2, 1/2]^d$ be the unit square in $\mathbb{R}^d$ and define the N dimensional matrix

$$h_{\mathbf{k},b}(j,j') := \sum_{\gamma\in\mathbb{Z}^d} e^{-i2\pi\mathbf{k}\cdot\gamma}\,\widetilde{T_b}(\gamma,j;0,j'), \quad \mathbf{k}\in\Omega.$$

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We then have:

$$\langle \tau_\gamma^b(w_{j,b}) | H_b \tau_{\gamma'}^b(w_{j',b}) \rangle = e^{ib\phi(\gamma,\gamma')} \int_\Omega e^{i2\pi \mathbf{k}\cdot(\gamma-\gamma')} h_{\mathbf{k},b}(j,j') d\mathbf{k}.$$

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It turns out that $h_{\mathbf{k},b}(j,j')$ has an asymptotic expansion in b, all its terms being real analytic in $\mathbf{k}$ and $\mathbb{Z}^d$-periodic. The spectrum of the matrix $h_{\mathbf{k},0}$ coincides with the N Bloch bands of H_0 corresponding to σ_0.

5. Assume that the magnetic field is slowly varying, i.e. it comes from $\mathbf{a}_\epsilon(\mathbf{x}) := \mathbf{a}(\epsilon\mathbf{x})$ with $\mathbf{a} \in [C^1(\mathbb{R}^2)]^2$ and $\sup_{\mathbf{x}\in\mathbb{R}^2} |\partial_j a_k(\mathbf{x})| \leq \mathrm{const}$ where

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Stokes theorem gives:

$$\phi_\epsilon(\mathbf{x},\mathbf{x}') = \int_{[0,\mathbf{x}']} \mathbf{a}_\epsilon + \int_{[\mathbf{x}',\mathbf{x}]} \mathbf{a}_\epsilon - \int_{[0,\mathbf{x}]} \mathbf{a}_\epsilon.$$

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Up to an error of order ϵ we have:

$$e^{i\int_{[\gamma',\gamma]} \mathbf{a}_\epsilon} \int_\Omega e^{i2\pi\mathbf{k}\cdot(\gamma-\gamma')} h_{\mathbf{k},0}(j,j') d\mathbf{k} \quad \text{in} \quad l^2(\mathbb{Z}^2)\otimes\mathbb{C}^N.$$

Peierls substitution

Up to an another error of order ϵ we have:

$$e^{i\mathbf{a}_\epsilon((\gamma+\gamma')/2)\cdot(\gamma-\gamma')}\int_\Omega e^{i2\pi\mathbf{k}\cdot(\gamma-\gamma')}h_{\mathbf{k},0}(j,j')d\mathbf{k} \quad \text{in} \quad l^2(\mathbb{Z}^2)\otimes\mathbb{C}^N.$$

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Consider the matrix valued symbol $F(\xi,\mathbf{x}) := h_{\xi-\mathbf{a}_\epsilon(\mathbf{x}),0}$. Every $\mathbf{x}\in\mathbb{R}^2$ can be written as $\gamma+\underline{x}$ with $\underline{x}\in\Omega$.

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The Schwartz integral kernel of F's Weyl quantization in $L^2(\mathbb{R}^2)\otimes\mathbb{C}^N \equiv [L^2(\Omega)\otimes l^2(\mathbb{Z}^2)]\otimes\mathbb{C}^N$ is:

$$\delta(\underline{x}-\underline{x}')e^{i\mathbf{a}_\epsilon(\underline{x}+(\gamma+\gamma')/2)\cdot(\gamma-\gamma')}\int_\Omega e^{i2\pi\mathbf{k}\cdot(\gamma-\gamma')}h_{\mathbf{k},0}(j,j')d\mathbf{k}.$$

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"Isospectrality up to order ϵ"

Perturbation of rational fluxes

Let $b = 2\pi \frac{p}{q} + \epsilon$ with $p, q \in \mathbb{N}$.
Denote by:

$$\Lambda_q := (q\mathbb{Z}) \times \mathbb{Z} = \{[q\gamma_1, \gamma_2] : \gamma_{1,2} \in \mathbb{Z}\}, \quad \mathcal{B}_q := \{[0,0], ..., [q-1, 0]\}.$$

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Every point $\gamma \in \mathbb{Z}^d$ can be uniquely represented as:

$$\alpha + \underline{x} = [q\gamma_1, \gamma_2] + \underline{x}, \quad \alpha \in \Lambda_q, \quad \underline{x} \in \mathcal{B}_q.$$

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The kernel of the effective operator can be re-expressed in terms of the new coordinates as follows:

$$H_b(\alpha, \underline{x}, j; \alpha', \underline{x}', j') = e^{ib\phi(\alpha+\underline{x}, \alpha'+\underline{x}')} \mathcal{T}(\alpha - \alpha' + \underline{x} - \underline{x}'; j, j').$$

Perturbation of rational fluxes

If

$$[U_b f](\alpha, \underline{x}, j) := e^{i\pi p\gamma_1\gamma_2} e^{ib\phi(\alpha,\underline{x})} f(\alpha, \underline{x}, j)$$

then

$$[U_b H_b U_b^*](\alpha, \underline{x}, j; \alpha', \underline{x}', j') = e^{i\epsilon\phi(\alpha,\alpha')}(-1)^{p(\gamma_1-\gamma_1')(\gamma_2-\gamma_2')} e^{ib\phi(\alpha-\alpha',\underline{x}+\underline{x}')} \cdot \mathcal{T}(\alpha - \alpha' + \underline{x} - \underline{x}'; j, j').$$

Perturbation of rational fluxes

This operator can be seen as an operator in $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^{qN}$ with the kernel:

$$\begin{aligned}\mathcal{H}_\epsilon(\gamma, \underline{x}, j; \gamma', \underline{x}', j') :=& e^{i\epsilon q \phi(\gamma,\gamma')} \\ & \cdot (-1)^{p(\gamma_1-\gamma_1')(\gamma_2-\gamma_2')} e^{i(b_0+\epsilon)(\gamma_2-\gamma_2')(\underline{x}_1+\underline{x}_1')/2} \\ & \cdot \mathcal{T}([q(\gamma_1-\gamma_1'), \gamma_2-\gamma_2'] + \underline{x} - \underline{x}'; j, j').\end{aligned}$$

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The new Bloch fiber matrix will be of the type $(Nq) \times (Nq)$ and equals:

$$\begin{aligned}h_{\mathbf{k},\epsilon}(\underline{x}, j; \underline{x}', j') = \sum_{\gamma \in \mathbb{Z}^d} & e^{-i2\pi \mathbf{k}\cdot\gamma} (-1)^{p\gamma_1\gamma_2} e^{i(\pi p/q+\epsilon/2)\gamma_2(\underline{x}_1+\underline{x}_1')} \\ & \cdot \mathcal{T}([q\gamma_1, \gamma_2] + \underline{x} - \underline{x}'; j, j').\end{aligned}$$

Harper model with half-flux

Here $d = 2$, $N = 1$, $\mathcal{T}(m, n) = 1$ if $m^2 + n^2 = 1$ otherwise it equals zero, and $b_0 = \pi$ i.e. $p = 1$ and $q = 2$.

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The Bloch matrix is of the type 2×2. Up to an ϵ order error, the new Bloch matrix is:

$$\left[\begin{array}{cc} 2\cos(2\pi k_2) & 2\cos(2\pi k_1) \\ 2\cos(2\pi k_1) & -2\cos(2\pi k_2) \end{array} \right].$$

Its two eigenvalues are given by:

$$\pm 2\sqrt{\cos^2(2\pi k_1) + \cos^2(2\pi k_2)}$$

which generate four Dirac points at $[\pm 1/4, \pm 1/4]$.

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Helffer-Sjöstrand and Bellissard shown that gaps of order $\sqrt{\epsilon}$ open around 0.

The second problem

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Consider the Hilbert space $L^2(\mathbb{R}^d)$ with $d \geq 2$. Let $\langle x \rangle := \sqrt{1 + |\mathbf{x}|^2}$ and let $\alpha \geq 0$.
We consider bounded integral operators $T \in B(L^2(\mathbb{R}^d))$ to which we can associate a locally integrable kernel $T(\mathbf{x}, \mathbf{x}')$ which is continuous outside the diagonal and obeys the following weighted Schur-Holmgren estimate:

$$||T||_\alpha :=$$
$$\max\left\{\sup_{\mathbf{x}' \in \mathbb{R}^d} \int_{\mathbb{R}^d} |T(\mathbf{x}, \mathbf{x}')| \langle \mathbf{x} - \mathbf{x}' \rangle^\alpha d\mathbf{x},\ \sup_{\mathbf{x} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |T(\mathbf{x}, \mathbf{x}')| \langle \mathbf{x} - \mathbf{x}' \rangle^\alpha d\mathbf{x}'\right\}.$$

Let us denote the set of all these operators with $\mathcal{C}^\alpha$.

If $T \in \mathcal{C}^\alpha$, we define $\{T_\epsilon\}_{\epsilon\in\mathbb{R}} \subset \mathcal{C}^\alpha$ given by the kernels

$$e^{i\epsilon\varphi(\mathbf{x},\mathbf{x}')}T(\mathbf{x},\mathbf{x}').$$

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The Hausdorff distance between two real compact sets A and B is defined as:

$$d_H(A,B) := \max\left\{ \sup_{x\in A} \inf_{y\in B} |x-y|,\ \sup_{y\in B} \inf_{x\in A} |x-y| \right\}.$$

If $T \in \mathcal{C}^\alpha$, we define $\{T_\epsilon\}_{\epsilon\in\mathbb{R}} \subset \mathcal{C}^\alpha$ given by the kernels

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Question: how regular is the following map?

$$\mathbb{R} \ni \epsilon \mapsto d_H(\sigma(T_\epsilon),\sigma(T)) \in \mathbb{R}_+$$

The second problem

Theorem

[H.C. and Purice 2011]. *Let* $H \in \mathcal{C}^\alpha$ *with* $\alpha > 0$ *be self-adjoint and consider a family of Harper-like operators* $\{T_\epsilon\}_{\epsilon \in \mathbb{R}}$ *as above. The map*

$$\mathbb{R} \ni \epsilon \mapsto d_H(\sigma(T_\epsilon), \sigma(T)) \in \mathbb{R}_+$$

is Hölder continuous with exponent $\beta := \min\{1/2, \alpha/2\}$*. More precisely, for all* ϵ_0 *we can find a numerical constant* $C_\beta > 0$ *such that:*

$$d_H(\sigma(T_{\epsilon_0+\delta}), \sigma(T_{\epsilon_0})) \leq C_\beta \, ||T||_{2\beta} \, |\delta|^\beta.$$

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Previous contributors: Elliot, Avron, Herbst, Simon, Helffer, Sjöstrand, Nenciu, Bellissard, Măntoiu, Iftimie,...

The third problem

Denote by $\mathcal{E}(\epsilon)$ one of the quantities $\sup \sigma(T_\epsilon)$, $\inf \sigma(T_\epsilon)$ or $||T_\epsilon||$, where T_ϵ is as before.

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What about the general case?

Theorem

[H.C. and Purice 2014].
If $1 \leq \alpha < 2$, then there exists a numerical constant $C_\alpha > 0$ with $\lim_{\alpha \nearrow 2} C_\alpha = \infty$, such that

$$|\mathcal{E}(\epsilon) - \mathcal{E}(0)| \leq C_\alpha \|T\|_\alpha |\epsilon|^{\alpha/2};$$

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If $1 \leq \alpha < 2$, *then there exists a numerical constant* $C_\alpha > 0$ *with* $\lim_{\alpha \nearrow 2} C_\alpha = \infty$, *such that*

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If $\alpha \geq 2$, *then there exists a numerical constant* $C > 0$ *such that*

$$|\mathcal{E}(\epsilon) - \mathcal{E}(0)| \;\leq\; C \|T\|_2 \, |\epsilon| \, \ln(1/|\epsilon|);$$

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Let $\alpha \geq 2$ *and assume that the magnetic field perturbation comes from a constant magnetic field. Then there exists a numerical constant* $C > 0$ *such that*

$$|\mathcal{E}(\epsilon) - \mathcal{E}(0)| \leq C \|T\|_2 \, |\epsilon|.$$

The fourth problem

Consider a slowly varying magnetic field $B_{\epsilon,\eta}(\mathbf{x}) := \epsilon(1+\eta b(\epsilon\mathbf{x}))$ and the corresponding magnetic matrix

$$e^{i\phi_{\epsilon,\eta}(\gamma,\gamma')} \int_\Omega e^{i2\pi\mathbf{k}\cdot(\gamma-\gamma')}\lambda(\mathbf{k})d\mathbf{k}.$$

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Question 2: if $\eta \neq 0$ is small, are the gaps preserved?

Ongoing work with Helffer and Purice.

Comments on the bibliography

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Existence and construction of localized Wannier functions:

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1. Fiorenza,D., Monaco,D., Panati,G.: Construction of real-valued localized composite Wannier functions for insulators. Preprint 2014 http://arxiv.org/abs/1408.0527

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2. When $N = 1$: Nenciu, Helffer-Sjöstrand.
3. If $d \leq 2$, ongoing work H.C., I. Herbst and G. Nenciu.

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Magnetic pseudo-differential calculus / Gauge covariant magnetic perturbation theory:

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2. Briet, P., H.C.: Locating the spectrum for magnetic Schrödinger and Dirac operators. Comm. P.D.E. **27**(5-6), 1079–1101 (2002)

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2. Briet, P., H.C.: Locating the spectrum for magnetic Schrödinger and Dirac operators. Comm. P.D.E. **27**(5-6), 1079–1101 (2002)
3. Măntoiu, M., Purice, R.: The magnetic Weyl calculus. J. Math. Phys. **45** (4), 1394–1417 (2004)

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1. Nenciu, G.: On asymptotic perturbation theory for quantum mechanics: Almost invariant subspaces and gauge invariant magnetic perturbation theory. J. Math. Phys. **43**(3), 1273-1298 (2002)
2. Briet, P., H.C.: Locating the spectrum for magnetic Schrödinger and Dirac operators. Comm. P.D.E. **27**(5-6), 1079–1101 (2002)
3. Măntoiu, M., Purice, R.: The magnetic Weyl calculus. J. Math. Phys. **45** (4), 1394–1417 (2004)
4. Iftimie, V., Măntoiu, M., Purice, R.: Commutator Criteria for magnetic pseudodifferential operators. Comm. P.D.E. **35**, 1058–1094 (2010)

Effective band Hamiltonians at weak magnetic fields:

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1. Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et équation de Harper. Springer Lecture Notes in Phys. **345**, 118-197 (1989)

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1. Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et équation de Harper. Springer Lecture Notes in Phys. **345**, 118-197 (1989)
2. Helffer, B., Sjöstrand, J.: Analyse semi-classique pour l'équation de Harper. II. Comportement semi-classique près d'un rationnel. Mém. Soc. Math. France (N.S.) **40**, 1-139 (1990)

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3. Helffer, B., Sjöstrand, J.: Semiclassical analysis for Harper's equation. III. Cantor structure of the spectrum. Mém. Soc. Math. France (N.S.) **39**, 1-124 (1989)

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3. Helffer, B., Sjöstrand, J.: Semiclassical analysis for Harper's equation. III. Cantor structure of the spectrum. Mém. Soc. Math. France (N.S.) **39**, 1-124 (1989)
4. Nenciu, G.: Dynamics of band electrons in electric and magnetic fields: rigorous justification of the effective Hamiltonians. Rev. Mod. Phys. **63**(1), 91-127 (1991)

Comments on the bibliography

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1. Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et équation de Harper. Springer Lecture Notes in Phys. **345**, 118-197 (1989)
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4. Nenciu, G.: Dynamics of band electrons in electric and magnetic fields: rigorous justification of the effective Hamiltonians. Rev. Mod. Phys. **63**(1), 91-127 (1991)
5. H.C., Purice, R., Iftimie, V.: ongoing work.

Spectral stability:

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1. Elliott, G.: Gaps in the spectrum of an almost periodic Schrodinger operator. C.R. Math. Rep. Acad. Sci. Canada **4**, 255-259 (1982)

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2. Avron, J.E., Simon, B.: Stability of gaps for periodic potentials under variation of a magnetic field. J. Phys. A: Math. Gen. **18**, 2199-2205 (1985)

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3. Bellissard, J.: Le papillon de Hofstadter. Séminaire Bourbaki **34**, 7-39 (1991-1992)

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4. Bellissard, J., Kreft, C., Seiler, R.: Analysis of the spectrum of a particle on a triangular lattice with two magnetic fluxes by algebraic and numerical methods. J. of Phys. A **24**(10), 2329-2353 (1991)

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5. H.C., Purice, R.: On the Regularity of the Hausdorff Distance Between Spectra of Perturbed Magnetic Hamiltonians. Operator Theory: Advances and Applications **224**, 55–66 (2012)

Comments on the bibliography

Lipschitz regularity of spectral edges:

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1. Bellissard, J.: Lipshitz Continuity of Gap Boundaries for Hofstadter-like Spectra. Commun. Math. Phys. **160**, 599-613 (1994)

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2. Kotani, M.: Lipschitz continuity of the spectra of the magnetic transition operators on a crystal lattice. J. Geom. Phys. **47** (2-3), 323–342 (2003)

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4. H.C.: On the Lipschitz Continuity of Spectral Bands of Harper-Like and Magnetic Schrödinger Operators. Annales Henri Poincaré **11**(5), 973–990 (2010)
5. H.C., Purice, R.: Spectral edge regularity of magnetic Hamiltonians, to appear in J. Lond. Math. Soc. (2015), http://arxiv.org/abs/1406.6624

Thank you!