Intersection of 4 quadrics in $\mathbb{C}P^6$ and Abelian varieties: An experimental study

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- Based on the results of F. Kötter, W.Barth, M.Adler, P. van Moerbeke, L.Heine, E. Horozov, et al

Intersection of 4 quadrics in $\mathbb{P}^6(x_1 : \cdots : x_6 : x_0)$ and Abelian varieties

A rank k quadric $Q(x) = \{\sum_{i,j=0}^{6} q_{ij}x_i x_j = 0\}$, k=rank q is a projective closure of an affine quadric $\bar{Q} \in \mathbb{C}^6(X_1, \ldots, X_6)$:

$$\sum_{i,j=1}^{6} q_{ij}X_iX_j + \cdots + \sum_{i=1}^{6} q_{i0}X_i = -q_{00}, \quad X_i = \frac{x_i}{x_0}$$

The space W of all quadrics in $\mathbb{P}^6$ admits stratification

$$W_6 \subset \cdots \subset \underbrace{W_2}_{\text{corank 2 quadrics}} \subset \underbrace{W_1}_{\text{corank 1 quadrics}} \subset W$$

For given four quadrics $Q_0(x), \ldots Q_3(x) \in \mathbb{P}^6$ consider their *linear system*

$$\Lambda = \{t_0 Q_0(x) + \cdots + t_3 Q_3(x) = 0\}, \quad (t_0 : t_1 : t_2 : t_3) = \mathbb{P}^3$$

with stratification $\cdots \subset \tilde{W}_2 \subset \tilde{W}_1 \subset \Lambda$, $\quad \tilde{W}_k = W_k \cap \Lambda$.
The discriminant surface $\widetilde{W}_1 = \{\, |\, t_0 q_0 + \cdots + t_3 q_3 \,| = 0\} \subset \Lambda$ defines the set of degenerate quadrics. For a general Λ, $\operatorname{codim} \tilde{W}_k = k$.
If this does not hold, e.g., $\tilde{W}_4 \subset \tilde{W}_3 \subset \tilde{W}_2 \subset \tilde{W}_1 \subset \Lambda$, Λ is *singular*.

Let $\mathcal{I}_c$ be intersection of 4 quadrics $Q_0(x), \ldots, Q_3(x)$ of the specific form[1]

$$\sum_{i=1}^{3} \left(a_i x_i^2 + a_{i+3} x_{x+3}^2 + 2b_{i,i+3} x_i x_{i+3} \right) = c\, x_0^2$$

and $\mathcal{S} \subset \mathcal{I}_c \cap \{x_0 = 0\}$ be its infinite part.

Theorem (Adler, van Moerbeke, 1988)

$\mathcal{I}_c \setminus \mathcal{S}$ is an open subset of a 2-dim. Abelian variety $\mathcal{A}$ iff the discriminant surface $\widetilde{W}_1$ contains
1) a union of lines (rational curves) of quadrics of rank ≤ 4, or
2) an irreducible non-degenerate elliptic curve $\mathcal{E}$ of quadrics of rank ≤ 4.

In the first case $\mathcal{A}$ is the Jacobian of a genus 2 curve,

In the second case $\mathcal{A}$ has polarization (1,2) and can be regarded as Prym$(C, \mathcal{E})$ *of a double covering $C \to \mathcal{E}$ (genus(C)=3) ramified at 4 points on $\mathcal{E}$ corresponding to the quadrics of rank ≤ 3.*

[1]This condition can be relaxed.

$\mathrm{Prym}(C, E)$ with polarization (1,2) is not a Jacobian variety. To get an algebraic description of $\mathrm{Prym}(C, E)$ we use

Theorem (Horozov, van Moerbeke)

There are exactly 6 different genus 2 curves related to $\mathrm{Prym}(C, E)$ *via 2:1 isogenies.*

$$
\begin{array}{ccccc}
\mathrm{Jac}(\widetilde{\Gamma}_1) & & \mathrm{Jac}(\widetilde{\Gamma}_2) & & \mathrm{Jac}(\widetilde{\Gamma}_3) \\
\| & \searrow & \downarrow & \swarrow & \| \\
\| & & \mathrm{Prym}(C, \sigma) & & \| \\
\Downarrow & \swarrow & \downarrow & \searrow & \Downarrow \\
\mathrm{Jac}(\Gamma_1) & & \mathrm{Jac}(\Gamma_2) & & \mathrm{Jac}(\Gamma_3) \\
\| & \searrow & \downarrow & \swarrow & \| \\
\| & & \mathrm{Prym}^*(C, \sigma) & & \| \\
\Downarrow & \swarrow & \downarrow & \searrow & \Downarrow \\
\mathrm{Jac}(\widetilde{\Gamma}_1) & & \mathrm{Jac}(\widetilde{\Gamma}_2) & & \mathrm{Jac}(\widetilde{\Gamma}_3)
\end{array}
$$

Given the curves C, E, explicit equations of all the six genus 2 curves $\Gamma_i, \widetilde{\Gamma}_i$ have been calculated in

V. Enolski, Yu. Fedorov. Experimental Mathematics, 2016

Ch. Ritzenthaler, M. Romagny arXiv:1612.07033, 2016

Example. The Clebsch integrable case on $e^*(3) = (K, p)$.

The Kirchhoff equations possess four independent quadratic integrals

$$
\begin{aligned}
Q_1 &= (K, K) - (c_2 + c_3)p_1^2 - (c_3 + c_1)p_2^2 - (c_1 + c_2)p_3^2\,,\\
Q_2 &= (p, p)\,,\\
Q_3 &= (K, p)\,,\\
Q_0 &= (K, DK) - \det D(p, D^{-1}p)\,, \qquad D = \mathrm{diag}(c_1, c_2, c_3).
\end{aligned}
$$

Set

$$
\begin{gathered}
Q_\alpha = h_\alpha, \quad Q_0 = h_0, \qquad h_\alpha, h_0 = \mathrm{const},\\
K_\alpha = x_\alpha/x_0\,, \quad p_\alpha = x_{\alpha+3}/x_0, \qquad \alpha = 1, 2, 3
\end{gathered}
$$

Then the integrals define the quadratic forms

$$
\begin{aligned}
\overline{Q}_1(x) &= x_1^2 + x_2^2 + x_3^2 - (c_2 + c_3)x_4^2 - (c_3 + c_1)x_5^2 - (c_1 + c_2)x_6^2 - h_1x_0^2\,,\\
\overline{Q}_2(x) &= x_4^2 + x_5^2 + x_6^2 - h_2x_0^2\,,\\
\overline{Q}_3(x) &= x_1x_4 + x_2x_5 + x_3x_6 - h_3x_0^2\,,\\
\overline{Q}_0(x) &= -c_1x_1^2 - c_2x_2^2 - c_3x_3^2 + c_2c_3x_4^2 + c_1c_3x_5^2 + c_1c_2x_6^2 + h_0x_0^2
\end{aligned}
$$

and the linear system $\Lambda_3 = \left\{ \sum_{k=0}^{3} t_k\overline{Q}_k(x) = 0 \right\}$.

The discriminant surface $\widetilde{W}_1 \subset \Lambda_3(t_0 : t_1 : t_2 : t_3)$ is

$$\begin{vmatrix} t_1\mathbf{I} - t_0 D & t_3\mathbf{I} & \mathbf{0} \\ t_3\mathbf{I} & t_0 D'' + t_2\mathbf{I} - t_1 D' & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & f(t) \end{vmatrix} = 0\,,$$

$$D' = \mathrm{diag}\,(c_2 + c_3, c_3 + c_1, c_1 + c_2)\,, \quad D'' = \mathrm{diag}\,(c_2 c_3, c_3 c_1, c_1 c_2)\,,$$
$$f(t) = -t_0 h_0 - t_1 h_1 - t_2 h_2 - t_3 h_3\,.$$

An unexpected result: $\widetilde{W}_1$ is a union of 3 cones and a plane:
$\widetilde{W}_1 = C_1 \cup C_2 \cup C_3 \cup H_d,$

$$C_\alpha = \left\{(t_1 - t_0 c_\alpha)(t_2 + c_\beta c_\gamma t_0 - (c_\beta + c_\gamma) t_1) = t_3^2\right\}\,, \quad (\alpha, \beta, \gamma) = (1, 2, 3),$$
$$H_d = \{t_0 h_0 + t_1 h_1 + t_2 h_2 + t_3 h_3 = 0\}\,.$$

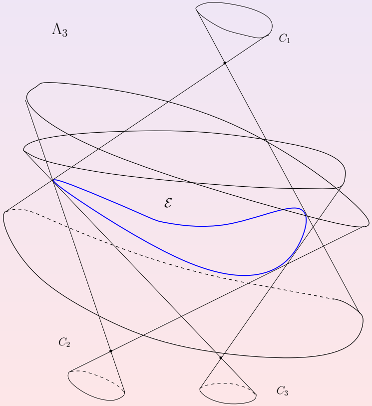
Moreover, $\mathcal{E} = C_1 \cap C_2 \cap C_3$ is not a finite number of points in Λ_3, as one might expect, but a spatial curve

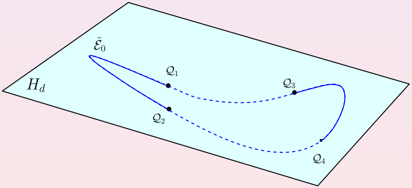
$$\mathcal{E} = \left\{t_2 = t_1^2 t_0\right\} \cap \left\{t_3^2 = (t_1 - c_1 t_0)(t_2 - (c_2 + c_3) t_1 + c_2 c_3 t_0)\right\}$$

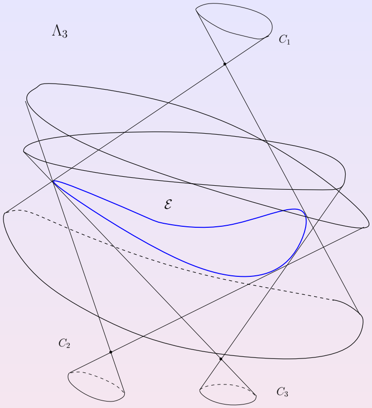
The one-to-one projection of $\mathcal{E}$ onto the plane $(t_1 : t_3 : t_0)$ is the elliptic curve

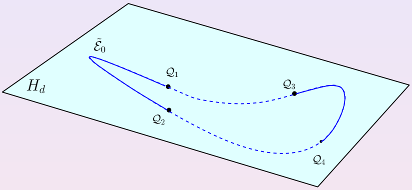
$$\mathcal{E}_0 = \left\{ t_3^2 = (t_1 - c_1)(t_1 - c_2)(t_1 - c_3) \right\} .$$

Along $\mathcal{E}$ the discriminant surface $\widetilde{W}_1$ has triple self-intersections, any point of $\mathcal{E} \subset \Lambda$ is a quadric in $\mathbb{P}^6$ of rank ≤ 4.
4 points of intersection of H_d with $\mathcal{E}$ correspond to quadrics of rank 3.









Theorem (L. Heine (1984))

A generic complex invariant manifold $\mathcal{I}_h$ of the Clebsch system is an open subset of $\mathrm{Prym}(\mathcal{C}, \mathcal{E})$ *related to the genus 3 non-hyperelliptic curve*

$$\mathcal{C} \; : \; \mu^2 = h_2 t^2 + h_1 t + h_0 + 2h_3 \sqrt{(t - c_1)(t - c_2)(t - c_3)}$$

How to find a parameterization of $\mathcal{I}_h \subset \mathbb{C}^6$?

Parametrization of $\mathcal{E}$:

$$\mathcal{E} = \left\{ t_0 = 1, \;\; t_1 = t, \;\; t_2 = t^2, \;\; t_3 = \sqrt{\Psi(t)} \right\} ,$$
$$\Psi(t) = (t - c_1)(t - c_2)(t - c_3) , \qquad t \in \mathbb{P}^1 .$$

Then we have family of quadrics in $\mathbb{P}^6$ of rank ≤ 4:

$$\mathcal{E} \, : \, \sum_{\alpha=1}^{3} \left(\sqrt{t - c_\alpha}\, x_\alpha + \frac{\sqrt{\Psi(t)}}{\sqrt{t - c_\alpha}} x_{\alpha+3} \right)^2 = \left(h_2 t^2 + h_1 t + h_0 + 2 h_3 \sqrt{\Psi(t)} \right) x_0^2 .$$

This family was first found by F. Kötter (1891), who used it to obtain explicit solutions of the Clebsch system:

Let $\{Q_1 = (s_1, \sqrt{\Psi(s_1)}), \dots, Q_4 = (s_1, \sqrt{\Psi(s_1)}) = \mathcal{E} \cap H_d$:

$$\sum_{\alpha=1}^{3} \left(\sqrt{s_j - c_\alpha}\, x_\alpha + \frac{\sqrt{\Psi(s_j)}}{\sqrt{s_j - c_\alpha}} x_{\alpha+3} \right)^2 = 0, \quad j = 1, 2, 3, 4$$

$$\sum_{\alpha=1}^{3}\left(\sqrt{s_j-c_\alpha}\,x_\alpha+\frac{\sqrt{\Psi(s_j)}}{\sqrt{s_j-c_\alpha}}x_{\alpha+3}\right)^2=0,\quad j=1,2,3,4$$

Under a linear change $(x_\alpha, x_{\alpha+3}) \to (\xi_\alpha, \eta_\alpha)$ the above rank 3 quadrics (only 3 of them are independent !) take the form

$$\sum_{\alpha=1}^{3}(\xi_\alpha+\eta_\alpha)^2=0 \quad \sum_{\alpha=1}^{3}(\xi_\alpha-\eta_\alpha)^2=0,$$

$$\left(d_\alpha\xi_\alpha+\frac{\eta_\alpha}{d_\alpha}\right)^2=0, \quad \sum_{\alpha=1}^{3}\left(d_\alpha\xi_\alpha-\frac{\eta_\alpha}{d_\alpha}\right)^2=0,$$

with some constants d_α. They give rise to the following 3 *independent* relations

$$\sum_{\alpha=1}^{3}(\xi_\alpha^2+\eta_\alpha^2)=0\,, \quad \sum_{\alpha=1}^{3}\left(d_\alpha^2\xi_\alpha^2+\frac{\eta_\alpha^2}{d_\alpha^2}\right)=0\,, \quad \sum_{\alpha=1}^{3}\xi_\alpha\eta_\alpha=0\,.$$

The latter defines a two-dimensional manifold in $\mathbb{P}^5=(\xi_1:\xi_2:\xi_3:\eta_1:\eta_2:\eta_3)$.

This manifold is isomorphic to the set $\mathcal{T}$ of common tangent lines ℓ of two confocal quadrics in $\mathbb{C}^3 = (X_1, X_2, X_3)$ given by

$$\mathcal{Q}_j = \left\{ \frac{X_1^2}{d_1^2 - \nu_j} + \frac{X_2^2}{d_2^2 - \nu_j} + \frac{X_3^2}{d_3^2 - \nu_j} = 1 \right\}, \quad j = 1, 2, \quad \nu_1 = 0\,,\ \nu_2 = d_1^2 d_2^2 d_3^2.$$

Following H. Knörrer, $\mathcal{T}$ is an 8-fold unramified covering of Jac(Γ),

$$\Gamma:\ w^2 = z(z - d_1^2)(z - d_2^2)(z - d_3^2)(z - d_1^2 d_2^2 d_3^2).$$

F. Kötter gave a parametrization of $\mathcal{T}$ in terms of 2 points $(z_1, w_1), (z_2, w_2)$ on Γ, which leads to parameterization in theta-functions of Γ with half-integer theta-characteristics

$$\eta_\alpha = L \sqrt{\frac{(z_1 - d_\alpha^2)(z_2 - d_\alpha^2)}{(d_\beta^2 - d_\alpha^2)(d_\gamma^2 - d_\alpha^2)}} = L\, C_\alpha \frac{\theta[\Delta + \delta_\alpha](u_1, u_2)}{\theta[\Delta](u_1, u_2)},$$

$$\xi_\alpha = \eta_\alpha \frac{\sqrt{z_1 z_2}}{d_1 d_2 d_3\, (z_1 - z_2)} \left(\frac{w_1}{z_1(z_1 - d_\alpha)} - \frac{w_2}{z_2(z_2 - d_\alpha)} \right)$$

$$= L\, D_\alpha \frac{\theta[\Delta + \delta_\alpha + \delta_0](u_1, u_2)}{\theta[\Delta](u_1, u_2)}, \qquad \alpha = 1, 2, 3.$$

Here the coordinates (u_1, u_2) on Jac(Γ) are the Abel image of the divisor $(z_1, w_1), (z_2, w_2)$, next $\Delta, \delta_\alpha, \delta_0$ are certain half-integer characteristics, and L is a nonzero factor (also depending on z_j, w_j). It can be calcualted by using one the quadratic integrals independent of the above rank 3

Theorem (F. Kötter (1891))

For certain values of $\mathfrak{a}, \mathfrak{b}$,

$$L^{-1} = \mathfrak{a}\,\eta_4 + \mathfrak{b}\,\eta_4 \,\frac{\sqrt{z_1 z_2}}{z_1 - z_2}\left(\frac{w_1}{z_1(z_1 - d_4^2)} - \frac{w_2}{z_2(z_2 - d_4^2)}\right),$$

$$\eta_4^2 = \frac{(z_1 - d_4^2)(z_2 - d_4^2)}{(d_4^2 - d_1^2)(d_4^2 - d_2^2)(d_4^2 - d_3^2)}, \qquad d_4 = d_1 d_2 d_3.$$

Then, for certain constants $\mathfrak{A}, \mathfrak{B}$,

$$L^{-1} = \mathfrak{A}\,\frac{\theta[\Delta + \delta_4](u_1, u_2)}{\theta[\Delta](u_1, u_2)} + \mathfrak{B}\,\frac{\theta[\Delta + \delta_0 + \delta_4](u_1, u_2)}{\theta[\Delta](u_1, u_2)}.$$

As a result, as $x_\alpha, x_{\alpha+3}$ *are linear combinations of* η_α, ξ_α, *one has*

$$x_\alpha = \frac{\mathfrak{A}_\alpha\,\theta[\Delta + \delta_\alpha](u_1, u_2) + \mathfrak{B}_\alpha\,\theta[\Delta + \delta_0 + \delta_\alpha](u_1, u_2)}{\mathfrak{A}\,\theta[\Delta + \delta_4](u_1, u_2) + \mathfrak{B}\,\theta[\Delta + \delta_0 + \delta_4](u_1, u_2)}$$

and similar expressions for $x_{\alpha+3}$.

HK discretization of the Clebsch case: Experiments with the 4 quadrics

$$\begin{cases} \tilde{M} - M = \epsilon(\tilde{M} \times AM + M \times A\tilde{M} + \tilde{P} \times AP + P \times A\tilde{P}), \\ \tilde{P} - P = \epsilon(\tilde{P} \times AM + P \times A\tilde{M}). \end{cases}$$

Fixing certain values of 4 constants of motion, one arrives at intersection of 4 quadrics in $\mathbb{C}^6(M, P)$ given by (*following A. Pfadler*)

$$Q_2 = -\frac{1392728800943552687\, {M_1}^2}{635041732453397568} + \frac{1081317358603017\,1 {M_2}^2}{529201443711164\,64}$$

$$-\frac{1392728800943552687\, M_3^2}{7990941800038586064} - \frac{15507960856265463687508220537\, {P_1}^2}{2185123035220551359200\,8}$$

$$+\frac{4587510103810042772196 91\, {P_2}^2}{174809842817644108736} - \frac{56456466843526296811882\,7 {P_3}^2}{174809842817644108736} + M_2\, P_2 = 0,$$

$$Q_3 = \frac{13249117652086408\,7 {M_1}^2}{488493640348767\,36} - \frac{1324911765208647\, M_2^2}{5658384667373292} + \frac{10813173586017\,1 M_3^2}{529201443711164}$$

$$+\frac{387699562943529715935475770\,5 P_1^2}{502867718370292448557\,8} - \frac{458770117103546642644526389\,9 P_2^2}{1609176698784935835384\,96}$$

$$+\frac{564609241670301016593462608\,9 P_3^2}{1609176698784935835384\,96} + M_3\, P_3 = 0,$$

$$
\begin{aligned}
Q_0 &= 1 - \frac{265468\, {P_1}^2}{1823} + \frac{9805\, P_2^2}{1823} - \frac{12071\, P_3^2}{1823} = 0, \\
Q_1 &= \frac{1081317358603017 1\, M_1^2}{5292014437111646 4} + \frac{38006418090887\, M_2^2}{1225983344597531} + \frac{3800641809088\, M_3^2}{13318236333397643} \\
&- \frac{620321500710913317110079\, P_1^2}{1012440339652188796429} + \frac{4581423701591780363415737 1\, P_2^2}{2024880679304377592859 4} \\
&- \frac{5642167592294337606543313409\, P_3^2}{20248806793043775928594} + M_1\, P_1 = 0.
\end{aligned}
$$

Consider the linear system of quadrics

$$
\Lambda_3 \ : \ t_1 Q_1 + t_2 Q_2 + t_3 Q_3 + t_0 Q_0 = 0.
$$

The discriminant surface $\tilde{W}_1 \subset \Lambda(t_0 : t_1 : t_2 : t_3)$ is the union of 3 cones C_1, C_2, C_3 and of the plane $\{t_0 = 0\}$.

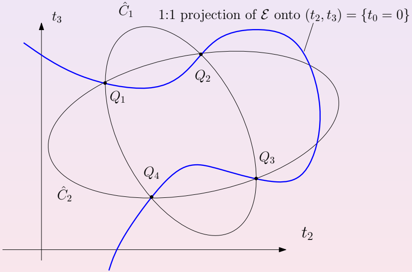
The intersection $C_1 \cap C_2 \cap C_3$ is a spatial elliptic curve $\mathcal{E}$. Thus, according to [AvM], the intersection of the 4 quadrics $Q_0, \ldots, Q_4$ is an Abelian variety.

Setting $t_1 = 1$, we obtain its affine part $E \subset \mathbb{C}^3(t_0, t_2, t_3)$. It is projected one-to-one (no branching) to the plane elliptic curve $\bar{\mathcal{E}}$ on the plane $\{t_0 = 0\}$ with the coordinates (t_2, t_3):

$$
\begin{aligned}
R(t_2, t_3) = 1 &- \frac{5640867029107772795 9}{7558440019602232212}\, t_2 + \frac{6711460476626436816 41}{8314284021562455433 2}\, t_3 \\
&- \frac{18873094535164528856161777467427675407 1}{47399423197482691225229801089369308 6}\, t_2^2 \\
&+ \frac{2454133011081014519779991967528879349}{2600078068978754318443763087732 82}\, t_2\, t_3 \\
&- \frac{177126923902147936335929401425978624423163 9}{11375861567395845894055152261448634064}\, t_2^2\, t_3 \\
&+ \frac{2099124413174301445219767958581493213781971}{11375861567395845894055152261448634064}\, t_2\, t_3^2 \\
&- \frac{12712815541250763955297950132998499083 17}{23699711598741345612614900544684654 3}\, t_3^2 \\
&+ \frac{26961739830389557454138593936714387569 1}{624018736554901036426503141055876 8}\, t_2^3 \\
&- \frac{447958276650272334055967487123161537311}{6240187365549010364265031410558768}\, t_3^3 = 0
\end{aligned}
$$

Applying MAPLE's command `Wierstrassform`, we get $\mathcal{E}$ in a canonical form

$$Y^2 = (X - E_1)(X - R_2)(X - E_3), \quad E_1 + E_2 + E_3 = 0,$$
$$E_1 = 25005378612380937262949172298855617466922914 6,$$
$$E_2 = -12372177593229544717316016307415267466443073 7,$$
$$E_3 = -12633201019151392545633155991440350000479840 9.$$



The 4 intersection points of $\mathcal{E}$ with $\{t_0 = 0\}$ are (up to 10^{-8})

$(t_2 = -1.49051153,\ t_3 = -1.28056092)$, $(t_2 = -1.28977159,\ t_3 = -1.11826304)$, $(t_2 = -0.35655629,\ t_3 = -0.2487487)$, $(t_2 = 0.57166862,\ t_3 = 0.605708301)$

and $t_0 = 0, t_1 = 1$.

Substituting these values into $t_1 Q_1 + t_2 Q_2 + t_3 Q_3 + t_0 Q_0 = 0$, as in the Clebsch case, we get 4 integrals as rank 3 quadrics in $\mathbb{P}^6$:

$$
\begin{aligned}
G_1 &= (0.05192301994\, M_1 + 96.29330714\, P_1)^2 \\
&\quad + (0.0401571127\, i\, M_2 + 18.5583836\, i\, P_2)^2 \\
&\quad + (0.031234934\, M_3 - 20.398727\, P_3)^2 = 0, \\
G_2 &= (0.00556740514\, M_1 - 89.8099126\, P_1)^2 \\
&\quad + (0.03745025367\, i\, M_2 - 17.21965624\, i\, P_2)^2 \\
&\quad + (0.02913249\, M_3 + 19.19255819\, P_3)^2 = 0, \\
G_3 &= (0.5582455209\, M_1 + 0.8952094727\, P_1)^2 \\
&\quad + (0.107287088\, i\, M_2 - 1.661683628\, i\, P_2)^2 \\
&\quad + (0.11904002\, M_3 + 1.04479376\, P_3)^2 = 0, \\
G_4 &= (0.7703328105\, M_1 + 0.6448613906\, P_1)^2 \\
&\quad + (0.1480458476\, i\, M_2 - 1.930706863\, i\, P_2)^2 \\
&\quad + (0.1642646049\, M_3 + 1.843689236\, P_3)^2 = 0
\end{aligned}
$$

with $i = \sqrt{-1}$.

The quadrics can be written (not uniquely !) in the "canonical" form

$$\sum_{\alpha=1}^{3} (\xi_\alpha + \eta_\alpha)^2 = 0\,, \quad \sum_{\alpha=1}^{3} \left(d_\alpha \xi_\alpha + \frac{\eta_\alpha}{d_\alpha} \right)^2 = 0\,, \quad \sum_{\alpha=1}^{3} \xi_\alpha \eta_\alpha = 0\,,$$

with $d_1 = \pm 212.8614773, \quad d_2 = \pm 5.68386016, \quad d_3 = \pm 8.10758865$
under the linear change

$$M_1 = 0.8418528529\,\xi_1 - 2.703246092\,\eta_1, \quad P_1 = 2.210698954\,\eta_1,$$
$$M_2 = -4.380447054\,i\,\xi_2 + 0.3757181878\,i\,\eta_2, \quad P_2 = -0.3070821200\,i\,\eta_2,$$
$$M_3 = 3.947949654\,\xi_3 + 0.4830243884\,\eta_3, \quad P_3 = -0.3947806768\,\eta_3.$$

In fact, P_i also include ξ_i but with coefficients of order 10^{-5}.
Then, up to a factor $L(u_1, u_2)$ the new coordinates ξ_α, η_α admit parameterization

$$\eta_\alpha = L(u)\, C_\alpha \frac{\theta[\Delta + \delta_\alpha](u_1, u_2)}{\theta[\Delta](u_1, u_2)}, \quad \xi_\alpha = \chi(u)\, D_\alpha \frac{\theta[\Delta + \kappa_\alpha](u_1, u_2)}{\theta[\Delta](u_1, u_2)},$$

in theta-functions of $\Gamma : \; w^2 = z(z - d_1^2)(z - d_2^2)(z - d_3^2)(z - d_1^2 d_2^2 d_3^2)$.
The factor $L(u_1, u_2)$ is calculated uniquely by using the theorem of Kötter last integral,

. Thus, we get an explicit parameterization of the intersection of the 4 quadrics in the HK discretization of the Clebsch case.

Reconstruction of the covering $C \to \mathcal{E}$ ramified at $Q_1, \dots, Q_4 \in \mathcal{E}$

There are 4 different coverings $C \to \mathcal{E}$ with the same ramification locus. Which one to choose ?

The coordinate t_0 has presisely 4 simple zeros at $Q_1, \dots, Q_4 \in \mathcal{E}$, and it is a rational function of t_2, t_3.

Parametrization of t_2, t_3, t_0 $(t_1 = 1)$ in terms of X, Y on the canonical elliptic curve $Y^2 = (X - E_1)(X - R_2)(X - E_3)$:

$$t_2 = \frac{F_2(X,Y)}{(X - E_1)\rho_2(x)}, \quad t_3 = \frac{F_3(X,Y)}{(X - E_1)\rho_2(x)},$$

$$\rho_2(X) = X^2 + 1756597211462590761736770442155111322135863845930 23\, X$$

$$+7383199311283652498718978761157297735817929862714430 7$$

$$302957955897103671043509081728738749160856174475479 3095$$

$$64274251240366535410244049012742632537162696160706 56,$$

$$t_0 = \frac{P_2(X)Y + P_4(X)}{\rho_2(X)\, R_4(X)},$$

where F_2, F_3, P_2, R_2 are certain polynomials of the corresponding degree.

$$t_0 = \frac{P_2(X)Y + P_4(X)}{\rho_2(X)\, R_4(X)}, \quad Y^2 = (X - E_1)(X - R_2)(X - E_3).$$

Proposition (Conjecture)

1) The intersection of the quadrics $Q_0, \ldots, Q_3$ is the 2-dim. Prym variety $\mathrm{Prym}(C, \mathcal{E})$, *where the genus 3 curve C is given by*

$$W^2 = P_2(X)Y + P_4(X), \quad Y^2 = (X - E_1)(X - R_2)(X - E_3).$$

2) $\mathrm{Prym}(C, \mathcal{E})$ *is a 8 -fold covering of* $\mathrm{Jac}(\Gamma)$

$$\Gamma: \; w^2 = z(z - d_1^2)(z - d_2^2)(z - d_3^2)(z - d_1^2 d_2^2 d_3^2),$$
$$d_1 = \pm 212.8614773, \quad d_2 = \pm 5.68386016, \quad d_3 = \pm 8.10758865.$$

Item (2) has been checked by calculating the equations of the genus 2 curves $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3$ and by showing that one of them is birationally equivalent to the above Γ.