5p : Ramification of crystalline representations

§1 Introduction

Notation k: perfect field of char $= p > 0$

$W := W(k) \circlearrowleft \sigma : (a_0, a_1, \dots) \mapsto (a_0^p, a_1^p, \dots)$

$\bar{K}_0$: alg. closure of $K_0 = \mathrm{Frac}(W)$.

$G_{K_0} = \mathrm{Gal}(\bar{K}_0/K_0)$ $\quad r \in \mathbb{Z}_{\geq 0}$

V: crystalline p-adic rep of G_{K_0} of Hodge-Tate weights $\subseteq [0, r]$

V: crys $\Leftrightarrow V^\vee$: crys

i.e. For $D = \mathrm{Hom}_{G_{K_0}}(V, B_{crys})$, $\in MF_{K_0}(\varphi)$,

$D_{crys}(V^\vee)$

$\dim_{K_0}(D) = \dim_{\mathbb{Q}_p}(V)$

& $gr^i D = 0$ unless $i \in [0, r]$

Aim

Study & classify

G_{K_0}-stable $\mathbb{Z}_p$-lattices $T \subseteq V$

& their quotients for $r < p-1$.

— "integral" p-adic Hodge theory.

e.g.

- $A_{/W}$ abelian scheme
- $T_pA := \varprojlim A[p^n](\bar{K}_0)$, $\quad V = V_pA := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_pA$

$\Rightarrow V$: crystalline of HT wts $\subseteq [0, 1]$

- $T = T_pA \subseteq V$ $\quad G_{K_0}$-stable $\mathbb{Z}_p$-lattice
- $A[p^n](\bar{K}) = T/p^nT$: its quotient

Strategy

classify $\mathbb{Z}_p$-lattices $T \subseteq V$ & their quotients

by "W-lattices" $M \subseteq D$
& their quotients
(Fontaine-Laffaille, 1982)

<u>Rem</u>

- semistable case
- general r

} are treated

by a more recent theory of

$(\mathfrak{S}, \hat{G})$-modules due to T. Liu (2008)

($\leftarrow$ based on Kisin's theory)

<u>Final goal of (5P)</u>

Abrashkin, Fontaine's ramification bound (~1993)

for subquotients killed by p

of such V

$\rightsquigarrow$ applied to prove non-existence

or rarity of varieties $/\mathbb{Q}$

w/ everywhere good reduction (6P)

- $X/\mathbb{Q}$ proper sm. everywhere good red.

$\Rightarrow$ Fm $i \neq j,\ i+j \leq 3$, $H^j(X, \Omega_X^i) = 0$

vanishing of Hodge numbers

§2 Fontaine-Laffaille modules

Fontaine-Laffaille
Ann. Sci. ENS (1982)

Def.

- a Fontaine-Laffaille module $/W$ is a triplet

$$\mathcal{M} = (M, \{M^i\}_{i\in\mathbb{Z}}, \{\varphi^i\}_{i\in\mathbb{Z}}) \quad \text{with}$$

— a W-module M

— decreasing filtration

$M \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \supseteq \cdots$ by W-submodules

s.t. $M = \bigcup_{i\in\mathbb{Z}} M_i$ (exhaustive) & $\bigcap_{i\in\mathbb{Z}} M_i = 0$ (separated)

— σ-semilinear maps $\varphi^i_M : M^i \to M$

s.t.
$$\begin{array}{ccc} M_{i+1} & \xrightarrow{\varphi^{i+1}_M} & M \\ \cap & \circlearrowleft & \downarrow \times p \\ M_i & \xrightarrow[\varphi^i_M]{} & M \end{array}$$

- A morphism of FL module $/W$

$f: \mathcal{M} \to \mathcal{N}$ = $f: M \to N$ W-linear
s.t. $f(M^i) \subseteq N^i$
$f \circ \varphi^i_M = \varphi^i_N \circ f$ on M^i $(\forall i)$

- a sequence of FL modules $/W$

$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ is <u>exact</u> if

$0 \to L \to M \to N \to 0$ & $0 \to L^i \to M^i \to N^i \to 0$
(exact) for any i
(for underlying W-modules)

$\underline{MF}_W$: their (exact) category

- For $s \in \mathbb{Z}_{\geq 0}$,

$\underline{MF}^{f,[0,s]}_{W,\mathrm{tor}}$ is its full subcat of $\mathcal{M}$'s s.t.

— M : finite length W-module (tor)

— $M^0 = M$, $M^{s+1} = 0$ ([0,s])

— $\sum_{i\geq 0} \varphi^i(M^i) = M$ (f)

we also denote the cat dropping some of conditions
by dropping some of these symbols
say $\underline{MF}^f_W$

$\underline{MF}^{*\,k}_{k,k}$: p-torsion objects

Prop 1

(i) $\forall$ morphism $f : M \to N$ in $\underline{MF}^{f,[0,s]}_{W,tor}$ is strict w.r.t. filtrations

i.e. $f(M^i) = f(M) \cap N^i$

Coim Im

(ii) $\underline{MF}^{f,[0,s]}_{W,tor}$ is an Abelian & Artinian cat.

(e.g. $\mathrm{Coker}(f) = \left(N/f(M), \supseteq N^i/f(M^i), \varphi^i \right)$)

(iii) $M \in \underline{MF}^{f,[0,s]}_{W,tor} \Rightarrow M \supseteq M^i$ direct summand

pf For $M \in \underline{MF}^{[0,s]}_{tor}$

Define $\overline{M}$ by

$$0 \to \bigoplus_{i=1}^{s} M^i \xrightarrow{\theta_M} \bigoplus_{i=0}^{s} M^i \longrightarrow \overline{M} \to 0 \quad \text{(exact)}$$

$$\theta_M : (x_1, \dots x_s) \longmapsto (\overset{0}{x_1}, \overset{1}{-px_1 + x_2}, \overset{2}{-px_2 + x_3}, \dots, -px_s)$$

$\varphi_0 \quad \varphi_1$

inducing

$$\varphi_M : \overline{M} \longrightarrow M$$
$$\overline{(x_i)} \longmapsto \sum \varphi^i(x_i)$$

Then

$\lg(M) = \lg(\overline{M})$ &

$M \in \underline{MF}^{f,[0,s]}_{W,tor} \overset{\text{def}}{\Longleftrightarrow} \varphi_M :$ surj ($\Leftrightarrow$ inj $\Leftrightarrow$ bij)

~~$M \mapsto \overline{M}$ exact functor~~ ~~(Snake lem + right exact + length)~~

(i) For simplicity, assume f: inj.

Then

$$\begin{array}{ccc} \overline{M} & \longrightarrow & \overline{N} \\ \wr\downarrow & & \downarrow\wr \\ M & \hookrightarrow & N \end{array}$$

& $\overline{M} \hookrightarrow \overline{N}$ inj

Suppose $x \in (N^i \cap M) \setminus M^i$.

Take $j < i$ s.t. $x \in M^j \setminus M^{j+1}$ (M: p-power torsion)

(Replacing x by $p^l x$,) may assume $px \in M^{j+1}$

Put $y = (0, \dots 0, \overset{j}{x}, \overset{j+1}{-px}, 0 \dots 0) \in \bigoplus_{i=0}^{s} M^i$

Then $y \notin \mathrm{Im}\,\theta_M$ ($\because x \notin M^{j+1}$) & the image $\overline{y} \in \overline{M}$ is $\neq 0$.

$y \in \mathrm{Im}\,\theta_N$ & $\overline{y} = 0$ in $\overline{N}$ ($\because x \in N^i \subseteq N^{j+1}$)

Contradicting $\overline{M} \hookrightarrow \overline{N}$ inj.

(ii) follows from (i) & ☆ (Ker f satisfies / Coker f condition)

(iii) applying (i) to $M \xrightarrow{p^n} M$

$\Rightarrow \ p^n M^i = p^n M \cap M^i \ \Rightarrow \ M^i \subseteq M$ dir. summand

(ext gp calculation)

$\mathrm{Hom}(-, \mathcal{O}_K/(\pi))$?

- Cwah 未定 $\mathrm{Fil}^0(M \otimes A_{\mathrm{cris}})^{\phi_r = 1} \simeq T_{\mathrm{cris}}^*(M)^\vee(r)$
  for $M \in \underline{MF}^{f,[0,p-2]}_{\mathrm{tw,w}}$

Cwtravan にする主要点 :
- long exact seq of $\underset{\mathrm{Hom}}{\mathrm{Ext}}_{\mathrm{MF}}$
  → can study T_{cris}^* by using Yoneda ext

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$\underline{MF}^{f,[0,p-1]'}_{\text{tw-tor}}$ = (prime) the full subcat of $\underline{MF}^{f,[0,p-1]}_{\text{tw-tor}}$ of M's such that :

$\nexists \ M \twoheadrightarrow N$ non-zero quotient $\in \underline{MF}^{f,[0,p-1]}_{\text{tw-tor}}$ w/ $N^{p-1} = N$.

$\left(\underline{MF}^{f,[0,p-1]'}_{\text{tw-tor}} \cong \underline{MF}^{f,[0,p-2]}_{\text{tw-tor}} \right)$

§3 Galois representations

Aim: associate with $M \in \underline{MF}^{f,[0,p-1]}_{W,\mathrm{tor}}$ a G_{K_0}-rep $T^*_{crys}(M)$.

<u>Recall</u> $C := \widehat{\overline{K_0}}$ p-adic compl.

$$R = \varprojlim_{\mathrm{Frob}} \left(O_C/p \leftarrow O_C/p \leftarrow \cdots \right) \circlearrowleft G_{K_0}$$

$$\underline{p} = \left(-p, (-p)^{\frac{1}{p}}, (-p)^{\frac{1}{p^2}}, \dots \right) \in R$$

$\theta : W(R) \longrightarrow O_C$ natural surjection

w/ $\mathrm{Ker}\,\theta = (\xi)$, $\xi = p + [\underline{p}]$ $\circlearrowleft G_K$

$$A_{crys} := W(R)\left[\frac{\xi^n}{n!} \,\middle|\, n \geq 1 \right]^{\wedge} \quad \text{p-adic compl.}$$

$\cup$

$$\mathrm{Fil}^i A_{crys} := \overline{\left(\frac{\xi^n}{n!} \,\middle|\, n \geq i \right)} \quad \text{closure}$$

$R \longrightarrow R$, $x \longmapsto x^p$ induces

$\varphi : W(R) \to W(R)$

& $\varphi : A_{crys} \to A_{crys}$

Then,

∘ $$\varphi(\xi) = p + [\underline{p}^p] = p + (\xi - p)^p$$
$$= p\left(1 + (-1)^p p^{p-1} + \sum_{\ell=1}^{p-1} \frac{1}{p}\binom{p}{\ell} \xi^{\ell} (-p)^{p-\ell} + \frac{\xi^p}{p}\right)$$
$$= p\left(1 + \frac{\xi^p}{p} + p(\cdots)\right)$$

∘ Thus $\varphi(\mathrm{Fil}^i A_{crys}) \subseteq p^i A_{crys}$ for $\forall i \in \{0, 1, \dots, p-1\}$

∘ & putting $\varphi^i = \frac{1}{p^i}\varphi|_{\mathrm{Fil}^i A_{crys}}$, $\quad A_{crys}^i = \begin{cases} \mathrm{Fil}^i A_{crys} & (i \leq p-1) \\ 0 & (i = p) \end{cases}$

$$\left(A_{crys}, \overset{(0)}{A_{crys}} \supseteq \overset{(1)}{\mathrm{Fil}^1 A_{crys}} \supseteq \cdots \overset{(p-1)}{\mathrm{Fil}^{p-1} A_{crys}} \supseteq \overset{(p)}{0}, \{\varphi^i\}\right) \in \underline{MF}_W$$

∘ $A_{crys}/\mathrm{Fil}^i A_{crys}$: p-torsion free

& thus $\quad \left(A_{crys}/p^n\right)^i = \begin{cases} \mathrm{Fil}^i A_{crys}/p^n \mathrm{Fil}^i A_{crys} & (i < p) \\ 0 & (i = p) \end{cases}$

$$\left(\overset{(0)}{A_{crys}/p^n A_{crys}} \supseteq \overset{(1)}{\mathrm{Fil}^1 A_{crys}/p^n \mathrm{Fil}^1 A_{crys}} \supseteq \cdots \supseteq \overset{(p-1)}{\mathrm{Fil}^{p-1} A_{crys}/p^n \mathrm{Fil}^{p-1} A_{crys}} \supseteq \overset{(p)}{0}, \{\varphi^i\}\right)$$

$\in \underline{MF}_W$

∘ $A_{crys,\infty} := A_{crys} \otimes_W K_0/W \in \underline{MF}_W$ similarly.

For $M \in \underline{MF}^{f,[0,p-1]}_{W\text{-tor}}$ put

$$T^*_{crys}(M) = \mathrm{Hom}_{\underline{MF}_W}(M, A_{crys,\infty}) \curvearrowleft G_{K_0}$$

$\rightsquigarrow$ a functor $T^*_{crys} : \underline{MF}^{f,[0,p-1]}_{W\text{-tor}} \to \underline{Rep}_{\mathbb{Z}_p}(G_{K_0})$

Thm A

(1) The functor T^*_{crys} is exact & faithful

(2) $T^*_{crys}(M)$ is a fin. length $\mathbb{Z}_p$-module

& invariant factors of

$\left\{ \begin{array}{l} \text{the } \mathbb{Z}_p\text{-module } T^*_{crys}(M) \\ \quad W\text{-module } \quad M \end{array} \right.$

coincide

(3) The restriction

$T^*_{crys} : \underline{MF}^{f,[0,p-2]}_{W\text{-tor}} \to \underline{Rep}_{\mathbb{Z}_p}(G_{K_0})$ is full

(4) The essential image of (3)

is stable under subquotients.

1h 03m

free objects $r \in \{0, 1, \dots, p-2\}$ (fortement divisible)

$\underline{MF}_W^{fd,[0,r]} :=$ the full sub cat of $\underline{MF}_W$ of M's s.t.

- M: free W-module of fin rk
- $M \supseteq M^i$ direct summand for $\forall i$
- $M^0 = M$, $M^{r+1} = 0$
- $\sum_{i \geq 0} \varphi^i(M^i) = M$

For such M, define

$$T_{crys}^*(M) = \mathrm{Hom}_{\underline{MF}}(M, A_{crys}) \circlearrowleft G_{K_0}$$

$\Rightarrow$ By a $\varprojlim$ argument & Thm A-②,

$T_{crys}^*(M)$ is a free $\mathbb{Z}_p$-module of $rk = rk_W(M)$

$D := M[\frac{1}{p}]$ + induced fil, $\varphi = \varphi^0$

$\Rightarrow$ $D \in \underline{MF}_{K_0}(\varphi)$ & $V_{crys}^*(D) = T_{crys}^*(M)[\frac{1}{p}]$

$\left(\mathrm{Hom}_{\varphi, Fil, K_0}(D, B^+_{crys}) \simeq V_{crys}(D^\vee) \right)$

Conversely,

let V: crystalline G_{K_0}-rep of HT wts $[0, r]$

$D := D_{crys}^*(V) \in \underline{MF}_{K_0}(\varphi)$

A strongly divisible lattice in D is a W-submodule $M \subseteq D$ s.t.

- M: free of fin. rk w/ $D = M[\frac{1}{p}]$
- $\varphi(M \cap D^i) \subseteq p^i(M \cap D^i)$ for any i
- $\sum \frac{\varphi}{p^i}(M \cap D^i) = M$

$\Rightarrow$ $(M, M^i = M \cap D^i, \varphi^i = \frac{\varphi}{p^i}) \in \underline{MF}_W^{fd,[0,r]}$

& $D = M[\frac{1}{p}]$ as obj of $\underline{MF}_{K_0}(\varphi)$.

Thm B

① $M \in \underline{MF}_w^{fd,[0,r]} \Rightarrow T_{crys}^*(M)[\frac{1}{p}]$: crystalline of HT wts $\subseteq [0,r]$

② V & $D = D_{crys}^*(V)$ as above

Then T_{crys}^* induces an anti-equiv between

$$\begin{pmatrix} \text{the cat of} \\ \text{str. div.} \\ \text{lattices in } D \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{the cat. of} \\ G_{K_0}\text{-stable} \\ \mathbb{Z}_p\text{-lattices in } V \end{pmatrix}$$

$\Leftarrow$ $\varprojlim$ (Thm A)

+ Laffaille's Thm

" D : weakly admissible

$\Rightarrow \exists M \subseteq D$ str. div lattice "

a G_{K_0}-rep on a fin. length $\mathbb{Z}_p$-module $\overline{T}$

is said as <u>torsion crystalline of HT wts $\subseteq [0,r]$</u>

if $\exists V$: crystalline G_{K_0}-rep of HT wts $\subseteq [0,r]$

$\exists T' \subseteq T$ G_{K_0}-stable $\mathbb{Z}_p$-lattices in V

s.t. $\overline{T} \simeq T/T'$ as $\mathbb{Z}_p[G_{K_0}]$-modules

$\underline{Rep}_{tor}^{crys,[0,r]}(G_{K_0})$: their category

Fin.
$M \in \underline{MF}_{tor}^{f,[0,r]}$
$\exists$ resolution $0 \to L_1 \to L_2 \to M \to 0$ $L_i \in \underline{MF}_w^{fd,[0,r]}$
so $T_{crys}^*(M) \cong T_{crys}^*(L_1)/T_{crys}^*(L_2)$ tor crys
$\therefore$ well-def

well-def is 明らか
fully faithful : Thm A
ess surj : Thm B

Cor C

For $r \leq p-2$, T_{crys}^* induces an anti-equiv

$$\underline{MF}_{w,tor}^{f,[0,r]} \xrightarrow{\sim} \underline{Rep}_{tor}^{crys,[0,r]}(G_{K_0})$$

F5m

e.g. $\mathcal{G}/\mathcal{O}_{K_0}$ finite flat group scheme killed by some p-power

(e.g. $A[p^n]$ for an abelian scheme $A/\mathcal{O}_{K_0}$)

$\exists$ exact seq. $0 \to \mathcal{G} \to A \to B \to 0$

w/ A, B : abelian scheme $/\mathcal{O}_{K_0}$

& $0 \to T_pA \to T_pB \to \mathcal{G}(\overline{K}_0) \to 0$

i.e. G_{K_0}-rep $\mathcal{G}(\overline{K}_0)$ is $\in \underline{\mathrm{Rep}}^{\mathrm{crys},[0,1]}_{\mathrm{tor}}(G_{K_0})$

& it can be studied using $\underline{MF}^{f,[0,1]}_{\mathrm{tor}}$ if $p \neq 2$

__Rem__ Actually, for $p \neq 2$,

$\exists$ anti-equiv

$$\begin{pmatrix}\text{the cat} \\ \text{of } \mathcal{G}\text{'s}\end{pmatrix} \simeq \underline{MF}^{f,2}_{\mathrm{tor}}$$

cf. __Fontaine-Messing__

X/W proper sm

$\Rightarrow$ ① $H^m_{\mathrm{crys}}(X/W) := \varprojlim_n H^m_{\mathrm{crys}}(X/W_n, \mathcal{O}_{X/W_n})$

& $\varprojlim H^m_{\mathrm{crys}}(X/W_n, J^{[r]}_{X/W_n})$

+ crys Frob

$\in \underline{MF}^{fd,[0,m]}_{W}$ if $m<p$

② $H^m_{\mathrm{crys}}(X_n/W_n) \cong H^m_{\mathrm{crys}}(X_n/W_n, J^{[r]})$

+ crys Frob

$\in \underline{MF}^{f,[0,m]}_{W,\mathrm{tor}}$ if $m<p$

③ $m \leq p-2$

$\Rightarrow T_{\mathrm{crys},K}(H^m_{\mathrm{crys}}(X_n/W_n))$

$\wr|$

$H^m_{\mathrm{et}}(X_{\overline{K}_0}, \mathbb{Z}/p^n\mathbb{Z})$

w/ $T_{\mathrm{crys},K}(M)$

$= \mathrm{Fil}^0(A_{\mathrm{crys}} \otimes M)^{\varphi_m = 1}$

§3 Ramification theory

(cf. basic reference) Serre's "Corps Locaux"
Exposition / Fontaine, Il n'y a pas de variété abelienne sur $\mathbb{Z}$. (1985)

<u>Notation</u> :

- K : cdvf with <u>perfect</u> residue field k of char $= p > 0$
- $\mathcal{O}_K \supseteq \mathfrak{m}_K = (\pi_K)$

  $\overline{K} \supseteq K^{sep}$: alg / separable closure of K

- v_K : additive valuation normalized as $v_K(K^\times) = \mathbb{Z}$ & extended to $\overline{K}$
- $G_K := \mathrm{Gal}(K^{sep}/K)$

L/K : finite Galois ext in K^{sep}

<u>Want</u> to measure a distance of L/K from unramified extensions

<u>idea</u> : Define a closed normal subgp $G_K^{(u)} \subseteq G_K$ for $^\forall u \in \mathbb{R}_{>0}$ s.t.

L/K unram $\Leftrightarrow$ $G_L \supseteq G_K^{(u)}$ $(^\forall u > 0)$

- Then

$$u_{L/K} := \inf \{ u \in \mathbb{R}_{>0} \mid G_L \supseteq G_K^{(u)} \}$$

is a measure for the distance

Definition of $G_K^{(j)}$

Let L/K be a fin. Gal. ext in K^{sep}

with $\begin{cases} G = Gal(L/K) \\ \text{residue ext } \ell/k \\ e_{L/K} := v_L(\pi_K) = \dfrac{[L:K]}{[\ell:k]} \end{cases}$

Put

- For $g \in G$,

  - $\mathfrak{a}_g := (g(x) - x \mid x \in O_L) \underset{\text{ideal}}{\subseteq} O_L$

  - $i_{L/K}(g) := v_K(\mathfrak{a}_g)$

- For $i \in \mathbb{R}$,

$$G_{(i)} := \{ g \in G \mid i_{L/K}(g) \geq i \}$$

(usual def: $G_i := G_{(\frac{i+1}{e_{L/K}})}$)
(in Serre's book)

Then $\{G_{(i)}\}$: decreasing fil of G

s.t. $G_{(0)} = G$, $G_{(i)} = 0$ for $i \gg 0$

& $G_{(i)} = \mathrm{Ker}(G \to Gal(\ell/k))$

for $\forall\, 0 < i \leq \frac{1}{e_{L/K}}$: the inertia subgp

$G_{(\frac{1}{e_{L/K}})} = 0 \iff L/K$: unram.

but can't define " $G_{K,(i)} = \varprojlim_{L/K} Gal(L/K)_{(i)}$ "

since $Gal(L'/K) \twoheadrightarrow Gal(L/K)$ doesn't respect the fil $(\ )_{(i)}$.

<u>Solution</u> : shift the numbering

Put $\tilde{\varphi}_{L/K}(i) := \displaystyle\int_0^i \# G_{(x)}\, dx = $ (the slopes are equal)

$$= \sum_{g \in G} \min\{ i, i_{L/K}(g) \}$$

$\Rightarrow \tilde{\varphi}_{L/K} : \mathbb{R} \to \mathbb{R}$: piecewise linear, $\tilde{\varphi}_{L/K}(0) = 0$
& bijective

$\tilde{\psi}_{L/K} :=$ its inverse function &

put $G^{(u)} := G_{(\tilde{\psi}_{L/K}(u))}$ $\left(\begin{array}{l}\text{usual def:} \\ G^u = G^{(u+1)}\end{array}\right)$

Then $\mathrm{Gal}(L'/K)^{(u)} \twoheadrightarrow \mathrm{Gal}(L/K)^{(u)}$ &

we can define

$$G_K^{(u)} := \varprojlim_{L/K} \mathrm{Gal}(L/K)^{(u)}$$

s.t. $G_K^{(u)} \twoheadrightarrow \mathrm{Gal}(L/K)^{(u)}$ surj.

Def.

$$\left.\begin{cases} i_{L/K} := \sup\limits_{\substack{g\in G \\ g\neq 1}} i_{L/K}(g) \\ u_{L/K} := \tilde{\varphi}_{L/K}(i_{L/K}) \end{cases}\right\} \in \mathbb{R}_{\geq 0}$$

i.e. $$\begin{cases} i_{L/K} = \max\{ i \in \mathbb{R} \mid G_{(i)} \neq 0 \} \\ u_{L/K} = \max\{ u \in \mathbb{R} \mid G^{(u)} \neq 0 \} \\ \phantom{u_{L/K}} = \inf\{ u \in \mathbb{R} \mid G_L \supseteq G_K^{(u)} \} \end{cases}$$

properties

(a) $u_{L/K} = 0 \iff L/K$ unram

(b) $u_{L/K} = 1 \iff L/K$ tame

(i.e. $p \nmid e_{L/K}$)

~~(Indeed, $G_0/G_1 \hookrightarrow l^\times$ $g \mapsto \frac{g(\pi_L)}{\pi_L}$~~

~~$G_i/G_{i+1} \hookrightarrow l$ $g \mapsto \frac{g(\pi_L)}{\pi_L} - 1$ for $i \geq 1$)~~

(c) Take α s.t. $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ ($\Leftarrow$ k = perfect)

$f(X) \in \mathcal{O}_K[X]$ the minimal pol of α

$\beta \in \overline{K}$.

put $$\begin{cases} i = \max\limits_{g \in G} v_K(\beta - g\alpha) \\ u = v_K(f(\beta)). \end{cases}$$

Then $u = \tilde{\varphi}_{L/K}(i)$.

($\Leftarrow$ $u = \sum\limits_{g\in G} \min\{ i, i_{L/K}(g) \}$ by $u = \sum\limits_{g} v_K(\beta - g\alpha)$)

<u>different</u>

$\mathfrak{I}' = \{x \in L \mid x\mathfrak{I} \subseteq \mathcal{O}_L\}$

For L/K : fin. sep ext.

$\mathfrak{D}_{L/K}$: the different of L/K

$:= \{ x \in L \mid \mathrm{Tr}_{L/K}(x\mathcal{O}_L) \subseteq \mathcal{O}_K \}^{-1} \subseteq \mathcal{O}_L$

<u>properties</u>

(i) L/K unram $\Leftrightarrow$ $\mathfrak{D}_{L/K} = \mathcal{O}_L$

(ii) L/K tame $\Leftrightarrow$ $\mathfrak{D}_{L/K} = \mathfrak{m}_L^{e_{L/K}-1}$

(iii) $E/L/K$ fin. sep. exts

$$\Rightarrow \mathfrak{D}_{E/K} = \mathfrak{D}_{E/L} \cdot \mathfrak{D}_{L/K} \subseteq \mathcal{O}_E$$

(iv) $\delta_{L/K}$: the discriminant of L/K

$\left(:= \left(\det \left(\mathrm{Tr}_{L/K}(e_i e_j) \right)_{ij} \right) \right.$ for $\left. \mathcal{O}_L = \bigoplus \mathcal{O}_K e_i \right)$

$$\Rightarrow \delta_{L/K} = N_{L/K}(\mathfrak{D}_{L/K})$$

(v) L/K : totally ramified (i.e. $l = k$)

$(\Rightarrow \mathcal{O}_L = \mathcal{O}_K[\pi_L])$

$f(x) \in \mathcal{O}_K[x]$: the minimal pol. of π_L

$$\Rightarrow \mathfrak{D}_{L/K} = (f'(\pi_L))$$

(vi) L/K fin Gal. ext

$$\Rightarrow v_K(\mathfrak{D}_{L/K}) = u_{L/K} - i_{L/K} \quad \left(\Leftarrow (v) \right)$$

$\left(\mathrm{LHS} = \sum_{g \neq 1} i_{L/K}(g) \right)$

$(g = 1 \cdots e_{L/K})$

$$\left(\begin{array}{c} \sum_{g \neq 1} i_{L/K}(g) \\ \parallel \\ \sum_{g \neq 1} \min\{ i_{L/K}, i_{L/K}(g) \} \\ \parallel \\ u_{L/K} - i_{L/K} \end{array} \right)$$

Thm D (Fontaine '89, Abrashkin '90) Schémas propres et lisses sur $\mathbb{Z}$ (1993)

$r \in \{0,1,\dots,p-2\}$, $K = K_0$ as before

$M \in \underline{MF}_k^{f,[0,r]}$

L/K fin. ext s.t.

$$G_L = \mathrm{Ker}\left(G_K \to \mathrm{Aut}\left(T^*_{crys}(M)\right)\right)$$

Then $u_{L/K} \leqq 1 + \frac{r}{p-1}$.

In particular,

$$v_K(\mathcal{D}_{L/K}) < 1 + \frac{r}{p-1}$$

($i_{L/K} = 0 \Leftrightarrow L/K$ un $\Leftrightarrow u = 0$)

∗ p-power torsion case:

Abrashkin (invented clever new method recently)

Rem

- $r=0 \Rightarrow T^*_{crys}(M) = \mathrm{Hom}_\varphi(M, \bar{k})$

  $\Rightarrow$ This is unram & $u_{L/K} = 0$

  i.e. May assume $r \geqq 1$ & $p \neq 2$

- ~~$\exists$ generalizations~~
  - ~~$r \leqq p-1$ & M: p-power torsion: Abrashkin (1989–90)~~
  - ~~semistable case (2000~) for $\forall K$~~
    ~~Breuil-Messing, H., ($r \leqq p-2$)~~
    ~~Caruso-Liu, Caruso ($\forall r$)~~

- $\exists$ stronger variant for finite flat case & easier over $\mathcal{O}_K$ for $\forall K/\mathbb{Q}_p$:

  $\mathcal{G}/\mathcal{O}_K$ finite flat gp scheme killed by p^n (e.g. $\mathcal{G} = A[p^n]$ for $A/\mathcal{O}_K$ abelian scheme)

  $e_K := v_K(p)$, $G_L = \mathrm{Ker}(G_K \to \mathrm{Aut}(\mathcal{G}(\bar{K})))$

  $$\Rightarrow u_{L/K} \leqq e_K\left(n + \frac{1}{p-1}\right)$$ (Abrashkin, Fontaine '85)

A key lemma to bound ramification

L/K: fin. Gal ext, $m \in \mathbb{R}_{\geq 0}$

For any alg. ext E/K, put

$$\mathfrak{a}^m_{E/K} = \{x \in O_E \mid v_K(x) \geq m\}.$$

Consider the property (P_m) for L/K:

" For any alg. ext E/K, if

$\exists$ an O_K-alg hom

$\eta : O_L \to O_E / \mathfrak{a}^m_{E/K}$, then

$\exists$ K-alg hom $L \to E$ "

<u>Lemma</u> (Fontaine, M. Yoshida) ——

$$u_{L/K} = \inf\{m \in \mathbb{R}_{\geq 0} \mid (P_m) \text{ holds}\}$$

pf

① $m > u_{L/K} \Rightarrow (P_m)$ holds

Take α s.t. $O_L = O_K[\alpha]$ & its minimal pol $f(x)$

$\beta \in O_E$ s.t. $\beta \bmod \mathfrak{a}^m_{E/K} = \eta(\alpha)$

Then, $v_K(f(\beta)) \geq m > u_{L/K}$

$\underset{(c)}{\Rightarrow}$ $\exists\, \alpha_0$: conjugate of α s.t.

$$v_K(\beta - \alpha_0) > i_{L/K} = \sup_{g \neq 1} v_K(g(\alpha_0) - \alpha_0)$$

By Krasner's lemma, $\alpha_0 \in K(\beta)$

& $\exists\, L \hookrightarrow K(\beta) \subseteq E$.

② (P_m) holds $\Rightarrow m > u_{L/K} - \frac{1}{e_{L/K}}$

∵ shown by negating (P_m) for $m = u_{L/K} - \frac{1}{e_{L/K}}$

by explicitly constructing

a counterexample E/K

wlog assume L/K wild

then

$L \not\hookrightarrow E$

$\Downarrow$

$L \not\hookrightarrow E$

③ L'/L tame $\Rightarrow u_{L/K} = u_{L'/K}$, $e_{L'/K} = e_{L/K}\, e_{L'/L}$

Choosing arbitrarily large tame L'/L,

non-(P_m) for L'/K for $m = u_{L/K} - \frac{1}{e_{L'/K}}$ $\Rightarrow$ So is L/K □

§4 proof of main thms $\substack{A \\ D}$ ($K = K_0$)

Observation $\quad M \in \underline{MF}^{f_i\,[0,p-1]}_{W\text{-tor}}$

(i) $M \otimes_W W(\bar{k}) \in \underline{MF}^{f_i\,[0,p-1]}_{W(\bar{k}),\,\mathrm{tor}}$

& $T_{\mathrm{crys}}^*(M \otimes_W W(\bar{k})) \simeq T_{\mathrm{crys}}^*(M)\big|_{G_{K^{\mathrm{nr}}}}$

w/ K^{nr}: the max. unram ext of K in $\bar{K}$

(For module str & ramification of $T_{\mathrm{crys}}^*(M)$, we can reduce to the case of k = alg closed)

(ii) $R^{PD} := A_{\mathrm{crys}}/pA_{\mathrm{crys}}$

$$(\varprojlim(\mathcal{O}_{\bar{K}}/p \leftarrow \mathcal{O}_{\bar{K}}/p \leftarrow \cdots) \ni x = (x_0, \boxed{x_1}, x_2, \ldots))$$

$$\mathrm{pr}_1 : R \to \mathcal{O}_{\bar{K}}/p \quad \text{induces a } \vec{\sigma}\text{-semilinear,}$$

$$\underline{p} \longmapsto (-p)^{\frac{1}{p}}$$

G_K-equivar bijection of rings

$$(\mathcal{O}_{\bar{K}}/p)\frac{[Y_1, Y_2, \ldots]}{(Y_1^p, Y_2^p, \ldots)} \xrightarrow{\sim} R^{PD}$$

Sending $Y_\ell \longmapsto \overbrace{\delta \circ \cdots \circ \delta}^{\ell}(\underline{p})$

where
- $\delta(x) = \dfrac{x^p}{p}$ $\left(= (p-1)!\,\gamma_p(x)\right)$
- G_K-action on LHS:

$$g\left(\sum a_{\underline{\ell}} Y^{\underline{\ell}}\right) = \sum g(a_{\underline{\ell}}) Y^{\underline{\ell}}$$

$$(a_{\underline{\ell}} \in \mathcal{O}_{\bar{K}}/p)$$

Induced fil & φ^i on LHS are:

$$\begin{cases} \mathrm{Fil}^i(\mathrm{LHS}) = \left((-p)^{\frac{i}{p}},\ Y_\ell\ (\forall \ell \geq 1)\right) & (i \leq p-1) \\ \mathrm{Fil}^p(\mathrm{LHS}) = \left(Y_\ell\ (\forall \ell \geq 1)\right) \end{cases}$$

$$x = x_0 + \sum_{\ell \ge 1} x_\ell \gamma_\ell + (\text{higher terms})$$

$$\in \mathrm{Fil}^i(\mathrm{LHS}), \quad \begin{cases} x_0 \in (-p)^{\frac{i}{p}}(\mathcal{O}_{\bar K}/p) \\ x_\ell \in \mathcal{O}_{\bar K}/p \end{cases}$$

$$\Downarrow$$

$$\varphi^i(x) :=$$

- $\dfrac{\hat{x}_0^{\,p}}{(-p)^i}(1+\gamma_1)^i \qquad (i \le p-2)$

  for any lift $\hat{x}_0 \in \mathcal{O}_{\bar K}$ of x_0

- $\underline{\qquad\qquad} + x_1^p \quad (i = p-1,\ p \ge 3)$

- $\underline{\qquad\qquad} + \sum_{\ell \ge 1} x_\ell^p \quad (i = p-1,\ p = 2)$

(follows from $c(\xi) = 1 + \frac{\xi^p}{p} \mod p$)

$$A := (\mathcal{O}_{\bar K}/p)[\gamma_1]/(\gamma_1^p) \subseteq R^{PD}$$

+ induced fil & φ^i

$$\rightsquigarrow \begin{cases} (R^{PD}/A)^{p-1} = R^{PD}/A \\ \varphi^{p-1} = 0 \text{ on } R^{PD}/A \end{cases},$$

$M \in \underline{MF}_k^{f_i[0,p-1]}$ $\quad \sum \varphi^i(M^i) = M$

$\Downarrow$ — $\mathrm{Hom}(M, R^{PD}/A) = \mathrm{Ext}^1(M, R^{PD}/A) = 0$

$$\begin{cases} T_{\mathrm{crys}}^*(M) \simeq \mathrm{Hom}_{\underline{MF}_k}(M, A) \\ \mathrm{Ext}^1_{\underline{MF}_k}(M, R^{PD}) \simeq \mathrm{Ext}^1_{\underline{MF}_k}(M, A) \end{cases}$$

For any alg ext F/k,

- $A_F := \left(\mathcal{O}_F/p\mathcal{O}_F\right)^{[\Upsilon_1]}/(\Upsilon_1^p) \subseteq A = A_{\bar{k}}$

: induced fil & φ^i.

- For $M \in \underline{MF}_k^{f,[0,p-1]}$,

$$T^*_{\mathrm{crys},F}(M) := \mathrm{Hom}_{\underline{MF}}(M, A_F)$$

(iii) For $r \in \{1, \dots, p-2\}$,

- $b_F := \{ x \in \mathcal{O}_F \mid v_k(x) > \frac{r}{p-1} \}$

- $\mathcal{O}_F/b_F \in \underline{MF}_k$ by putting

$$\begin{cases} (\mathcal{O}_F/b_F)^i := \dfrac{\{ x \in \mathcal{O}_F \mid v_k(x) \geq \frac{i}{p} \}}{b_F} \\ \varphi^i(x) := \dfrac{\hat{x}^p}{(-p)^i} \bmod b_F \end{cases}$$

For $M \in \underline{MF}_k^{f,[0,r]}$,

$$A_F \twoheadrightarrow \mathcal{O}_F/b_F \qquad \Upsilon_1 \mapsto 0$$

induces

$$T^*_{\mathrm{crys},F}(M) \xrightarrow{\sim} \mathrm{Hom}_{\underline{MF}}(M, \mathcal{O}_F/b_F)$$

<u>pf</u>

- $x \in \overbrace{b_F + (\Upsilon_1)}^{\text{stable under } \varphi^i \ (\Leftarrow \underset{\sim}{r < p-1})}$

$$\Rightarrow \underbrace{\varphi^i \circ \cdots \circ \varphi^i}_{s}(x) = 0 \quad \text{for } s \gg 0, \ i \leq r$$

- Using this,

can show $\exists$ unique lift of

$f \in$ RHS to $M \to A_F$.

Simple objects / $k = \bar{k}$

$\left\{\begin{array}{l} A_{\inf} \xrightarrow{\text{replace}} O_{\bar{K}}/p \text{ or } A \\ T_{\text{cris}} \longrightarrow \text{polynomial} / W \end{array}\right.$

Suppose $k = \bar{k}$.

- For $h \in \mathbb{Z}_{\geq 1}$, let

  $i : \mathbb{Z}/h\mathbb{Z} \longrightarrow \mathbb{Z}$ be a map
  $n \longmapsto i_n$

  of period $= h$

  (i.e. $h = \min\{h' > 0 \mid i_{h'+n} = i_n \ (\forall n)\}$)

- $M(h;i) := \bigoplus_{m \in \mathbb{Z}/h\mathbb{Z}} k e_m$ with

$$\left\{\begin{array}{l} M(h;i)^{\ell} = \bigoplus_{i_m \geq \ell} k e_m \\ \varphi^{i_m}(e_m) = e_{m-1} \end{array}\right. \qquad \in \underline{MF}^{f}_{\bar{k}}$$

- For $\forall a \in \mathbb{F}_{p^h}$, define

  $[a] \in \mathrm{End}_{\underline{MF}_{\bar{k}}}(M(h;i))$ by

  $[a](e_m) = a^{p^{-m}} e_m$

$\Rightarrow$ Can show $e^{i_0}(e^{i_1}(\varphi^{i_{h-1}})) \cdots = x$ (use $k = \bar{k}$)

Lemma 2

(a) They are all the simple objects of $\underline{MF}^{f}_{\bar{k}}$

(b) $\underline{\mathrm{Hom}}_{\underline{MF}_{\bar{k}}}(M(h;i), M(h';i'))$

$$= \begin{cases} \mathbb{F}_{p^h} \text{ via } a \mapsto [a] & \text{if } h = h' \text{ \& } i = i' \text{ up to a shift} \\ 0 & \text{otherwise} \end{cases}$$

(c) $\mathrm{Ext}^1_{\underline{MF}_{\bar{k}}}(M, M(1, p-1)) = 0$ $(\forall M \in \underline{MF}^{f}_{\bar{k}})$

(d) $\underline{MF}^{f,[0,p-1]'}_{W,\mathrm{tor}}$: stable under subquot even for $k \neq \bar{k}$

(by (c) + $\varinjlim$ argument)

$\rightsquigarrow$ Thm A-④ mod. Thm A-①~③

prop 3 $(k=\bar{k})$

Suppose $\forall i_n \in [0, p-1]$. Then

(i) $\mathrm{Ext}^1_{\underline{MF}_{\bar{k}}}(M(h;i), R^{PD}) = 0$

(ii) via $a \mapsto [a]$,

$T_{\mathrm{cris}}^*(M(h;i))$ is an $\mathbb{F}_{p^h}$-vect sp

of dim $= 1$

(iii) $G_K \curvearrowright T_{\mathrm{cris}}^*(M(h;i))$ is given by

$\chi^{i_0+pi_1+\cdots+p^{h-1}i_{h-1}}$, where

$$\chi : G_K \longrightarrow \mathbb{F}_{p^h}^\times$$

$$g \longmapsto \frac{g(p^{\frac{1}{p^h-1}})}{p^{\frac{1}{p^h-1}}} \in \mu_{p^h-1}(\overline{\mathbb{F}_p}) = \mathbb{F}_{p^h}^\times$$

pf. may replace R^{PD} by A

∘ $0 \to A \to E \to M(h;i) \to 0$ (exact)

$\hat{e}_n \longmapsto e_n$: lift in E^{i_n}, $\hat{e}_n \in E$

$\Rightarrow \exists d_n \in A$ s.t. $\varphi^{i_n}(\hat{e}_n) = \hat{e}_{n-1} - d_n$

E splits

$\Leftrightarrow \exists u_n \in A^{i_n}$ s.t. $\varphi^{i_n}(\hat{e}_n + u_n) = \hat{e}_{n-1} + u_{n-1}$

i.e. $\varphi^{i_n}(u_n) - u_{n-1} = d_n$

∘ $(e_n \longmapsto u_n \in A^{i_n}) \in \mathrm{Hom}_{\underline{MF}_{\bar{k}}}(M(h;i), A)$

$\Leftrightarrow \quad \varphi^{i_n}(u_n) - u_{n-1} = 0$

By permuting & linearity, reduce to show:

$$(*)\begin{cases} u_n \in A^{i_n} \\ \begin{cases} \varphi^{i_n}(u_n) - u_{n-1} = 0 & (n \neq 1) \\ \varphi^{i_1}(u_1) - u_0 = d \end{cases}\end{cases}$$

$\begin{cases} \text{(i): has a solution for } \forall d \in A \\ \text{(ii): has exactly } p^h \text{ solutions if } d=0 \end{cases}$

Put $\begin{cases} u_n = a_n + b_n \tau_1 + \sum_{\ell \geq 2}^{p-1} c_{n,\ell} \tau_1^{\ell} \\ d = d_0 + d_1 \tau_1 + \sum_{\ell \geq 2}^{p-1} d_\ell \tau_1^{\ell} \end{cases}$

with $a_n \in (-p)^{\frac{i_n}{p}} (\mathcal{O}_{\overline{K}}/p) =: (\mathcal{O}_{\overline{K}}/p)^{i_n}$

& $\varepsilon(i) = \begin{cases} 0 & \text{if } i \neq p-1 \\ 1 & \quad i = p-1 \end{cases}$

Then

$$(*) \iff \begin{cases} a_n \in (\mathcal{O}_{\overline{K}}/p)^{i_n} \\ \quad\in \\ \dfrac{\hat{a}_n^p}{(-p)^{i_n}} (1+\tau_1)^{i_n} + \varepsilon(i_n) b_n^p - u_{n-1} = \begin{cases} 0 & (n \neq 1) \\ d & (n=1) \end{cases} \end{cases}$$

$\rightsquigarrow c_{n,\ell}$: determined by a_n, b_n, d_ℓ

$$\iff \left(\begin{cases} a_n \in (\mathcal{O}_{\overline{K}}/p)^{i_n} \\ \dfrac{\hat{a}_n^p}{(-p)^{i_n}} + \varepsilon(i_n) b_n^p - a_{n-1} = \begin{cases} 0 \\ d_0 \end{cases} \\ i_n \dfrac{\hat{a}_n^p}{(-p)^{i_n}} - b_{n-1} = \begin{cases} 0 \\ d_1 \end{cases} \end{cases}\right) \quad (\#)$$

<u>For $M \neq M(1, p-1)$:</u>

<u>(ii) & (iii)</u> $(d=0)$

- Can show $\varepsilon(i_n) b_n^p = 0 \ (\forall n)$ (by valuation calculation)
- any solution (a_n) of $(\#)$ uniquely lifts to a solution $\in \mathcal{O}_{\overline{K}}$ of

$$\frac{X_n^p}{(-p)^{i_n}} - X_{n-1} = 0 \quad (\forall n)$$

Σ

$$\mathrm{Hom}_{\underline{MF}_{k}}(M(h;i), R^{pp})$$
$$\parallel$$
$$\mathbb{F}_{p^h} \cdot \left(e_n \longmapsto (-p)^{\rho_n}(1+Y_1)^{i_{n+1}}\right)$$

w/ $\rho_n = \dfrac{i_{n+1} + p i_{n+2} + p^2 i_{n+3} + \cdots + p^{h-1} i_n}{p^h - 1}$

$(\Rightarrow (ii) \; (iii))$

(ie. reduce study of $T_{crys}^{*}(M)$
to finite flat alg /W
& its subrings / O_K)

(i)':

Construct a solution similarly from an equation $/W$

- $M = M(1, p-1)$: needs extra care.

($X^{p^2}+\cdots$ but and p get exactly right # of solutions)

(35)

~~pf of~~ Thm A-(1),(2) follows from Prop 3 by dévissage

dévissage (Ext's vanishing → exactness)

reducing $k=\bar{k}$, prop 3 (i) $\Big\} \Rightarrow$ exactness of T_{crys}^*

exactness, prop 3 (ii) $\Big\} \Rightarrow \lg_{\mathbb{Z}_p}(T_{\mathrm{crys}}^*(M)) = \lg_W(M)$

apply to $M[p^i]$ $\Rightarrow$ invariant factors are same.

$f: M \to N$, $M \searrow \mathrm{Inf} \nearrow N$ + A-② $\Rightarrow$ faithfulness

(41)

---

pf of Thm A-③ (fullness on $\underline{MF}_{\mathrm{tor}}^{f,[0,p-1]}$) 4

- May assume $k=\bar{k}$ by a descent
- $\mathrm{Hom}_{\underline{MF}_k}(M,N) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p[G_k]}(T_{\mathrm{crys}}^*(N), T_{\mathrm{crys}}^*(M))$

  : OK for M, N: simple.

- Enough to show

$$\mathrm{Ext}^1_{\underline{MF}_k}(M,N) \hookrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p[G_k]}(T_{\mathrm{crys}}^*(N), T_{\mathrm{crys}}^*(M))$$

  : inj

- May assume M, N: simple by dévissage

- Enough to show:

For $E \in \underline{MF}_k^{f,[0,p-1]'}$,

E: semisimple $\Leftrightarrow$ $G_K \curvearrowright T_{\mathrm{crys}}^*(E)$ tame.

(50)

Key lemma 4 ($k=\bar{k}$)

For $M \in \underline{MF}_{\bar{k}}^{fi[0,p-1]'}$ & any alg. ext F/K,

$G_F \curvearrowright T_{crys}^*(M)$ trivial

$\Leftrightarrow T_{crys,F}^*(M) = T_{crys}^*(M)$

$\underset{\mathrm{Hom}_{MF_k}(M, A_F)}{\parallel}$

pf $\Rightarrow$ induction on $\lg_W$:

$0 \to N \to M \to M(h_i i) \to 0$ (exact)

$\hat{e}_n \longmapsto e_n$

$\cap$

M^{i_n}

$\Rightarrow \varphi^{i_n}(\hat{e}_n) = \hat{e}_{n-1} + d_{n-1}$, ($d_{n-1} \in N$)

For $f \in T_{crys}^*(M)$,

put $f(\hat{e}_n) = a_n + b_n T_1 + \sum_{l \geq 2} c_{n,l} T_1^l$

induction hyp.

$\Rightarrow f(d_n) = \delta_n + \beta_n T_1 + \sum \gamma_{n,l} T_1^l \in A_F$

$\Downarrow$ $\varphi^{i_n} f(\hat{e}_n) = f(\varphi^{i_n}(\hat{e}_n))$

$$\begin{cases} \dfrac{\hat{a}_n^p}{(-p)^{i_n}} + \varepsilon(i_n) b_n^p - a_{n-1} = \delta_{n-1} \\[2ex] i_n \dfrac{\hat{a}_n^p}{(-p)^{i_n}} - b_{n-1} = \beta_{n-1} \end{cases}$$

$c_{n,l}$: determined by the others

- $\forall$ solution (a_n, b_n) uniquely lifts to a solution in $\mathcal{O}_{\bar{K}}$ of the equation $/\mathcal{O}_F$

$$\begin{cases} \dfrac{X_n^p}{(-p)^{i_n}} + \varepsilon(i_n) Y_n^p - X_{n-1} = \hat{\delta}_{n-1} \\[2ex] i_n \dfrac{X_n^p}{(-p)^{i_n}} - Y_{n-1} = \hat{\beta}_{n-1} \end{cases}$$

$\rightsquigarrow T_{crys}^*(M) = \mathrm{Im}\left((\text{Solutions in } \mathcal{O}_{\bar{K}})^{G_F} \right)$

$= \mathrm{Im}(\text{solutions in } \mathcal{O}_F) = T_{crys,F}^*(M)$ (1h) $\square$

- $K^{tr} :=$ the max. tamely ram. ext$/K$ in $\overline{K}$

For any $\rho \in \mathbb{Z}_{(p)}$,

$$\rho = \frac{a}{b} \quad \left(\begin{array}{l} a \in \mathbb{Z} \\ b \in \mathbb{Z}_{\geq 1} \end{array} \quad p \nmid b \right)$$

can choose $\pi_\rho \in K^{tr}$ s.t.

$$\begin{cases} \pi_\rho^b = (-p)^a \\ \pi_\rho \cdot \pi_{\rho'} = \pi_{\rho+\rho'} \end{cases}$$

- For any $\rho \in \mathbb{Z}_{(p)} \cap [0,1[$,

$\exists!\, h \in \mathbb{Z}_{\geq 1}$, $\exists!\, i : \mathbb{Z}/h\mathbb{Z} \to \{0,1,\dots,p-1\}$ of period h

s.t. $$\rho = \frac{i_1 + p i_2 + \cdots + p^{h-2} i_{h-1} + p^{h-1} i_0}{p^h - 1}$$

- Put $\overline{w}_\rho := \pi_{(-p)}^{\rho} (1+\tau_1)^{i_1}$ (actually choose these roots of $(-p)$ compatibly)

$$\in A_{K^{tr}}^{i_0}$$

$$\Rightarrow \varphi^{b}(\overline{w}_\rho) = \overline{w}_{p\rho - i_0} \in A_{K^{tr}}^{i_{h-1}}$$

$$p\rho - i_0 = \frac{i_0 + p i_1 + \cdots + p^{h-1} i_{h-1}}{p^h - 1}$$

$i_1 \to i_{h-1} \to i_{h-2} \to \cdots \to i_1$

$\rightsquigarrow M(h;i)$

put $A_{SS} := \mathrm{Span}_{\overline{k}} \{ \overline{w}_\rho \mid \rho \in \mathbb{Z}_{(p)} \cap [0,1[\}$

$\subseteq A_{K^{tr}}$: with induced str.

Then:

- $$A_{SS} \cong \bigoplus_{\substack{M:\text{ simple}/\cong \\ M \ncong M(1;p-1)}} M \quad (\rho<1)$$

- $\mathrm{Hom}_{\underline{MF}_{\overline{k}}}(M, A_{SS}) \hookrightarrow T^{K}_{crys}(M)$ ism

$\Leftrightarrow$ M: semisimple.

(1:05)

- $\left(A_K^{tr}/A_{ss}\right) \cong \bigoplus_{l=1}^{p-1} \left(\mathcal{O}_K^{tr}/p\right) \tau_1^{l}$ satisfies

$\rho p = \frac{p_1 + p p_1 + \cdots}{p^n - 1}$ & $\omega_{p\rho} = \pi_{\rho}^{p} + \pi^{0}$

$$\begin{cases} \cdot \ \left(A_K^{tr}/A_{ss}\right)^{p-1} = A_K^{tr}/A_{ss} \\ \cdot \ \varphi^{p-1} = 0 \end{cases}$$

$\left(\Leftarrow \varphi^{p-1}(b\tau_1) = b^p. \right.$ $\mathcal{O}_K^{tr}/p = \langle \pi_\rho \rangle_{\bar{k}}$, $\rho = \frac{i_1 + \cdots + i_n p^{n-1}}{p^n - 1}$

$$\begin{cases} \pi_{p\rho} = \omega_{p\rho} & \text{if } \rho < \frac{1}{p} \\ \pi_{p\rho} = 0 & \text{if } \rho \geq \frac{1}{p} \end{cases}$$

$\Downarrow$ — $\mathrm{Hom}(M, A_K^{tr}/A_{ss})$
$= 0$ (by $\Sigma\varphi^i(M) = M$)

- $\mathrm{Hom}_{\underline{MF}_{\bar{k}}}(M, A_{ss}) = T^{*}_{crys, K^{tr}}(M)$

Thus,

$G_K \curvearrowright T^{*}_{crys}(M)$ tame

$\Leftrightarrow \quad T^{*}_{crys, K^{tr}}(M) = T^{*}_{crys}(M)$

$\Leftrightarrow \quad M$: semi simple.

$\square$

L8

Pf of Thm B

① For $M \in \underline{MF}_w^{fd, [0,r]}$, put

$D := M[\frac{1}{p}]$, $V := V_{crys}^*(D) = T_{crys}$

By a dévissage for M/p^nM

from simple objects & $\varprojlim$,

can show

$M \hookrightarrow \mathrm{Hom}_{G_K}(T_{crys}^*(M), A_{crys})$

: inj

Thus

$\dim_{K_0} D \leq \dim_{K_0} D_{crys}^*(V)$

$\leq \dim_{\mathbb{Q}_p} V = \mathrm{rk}_W M$

$= \dim_{K_0} D$

& get $\begin{cases} V : \text{crystalline} \\ D = D_{crys}^*(V) \end{cases}$

② . Thm A $\overset{\varprojlim}{\Longrightarrow}$ fully faithful

. ess. surj : • Use Laffaille's thm

" D : weakly adm $\Rightarrow$ $\exists M \subset D$ str. div. lattice"

• For $T \subseteq V$, $\exists n_0$ s.t.

$p^{n_0}T / p^n T$ is a subquot of $T_{crys}^*(M)/p^{n+n_0}$ for $\forall n \geq n_0$

$\qquad \qquad T_{crys}^*(M/p^{n+n_0}M)$

& thus $\cong T_{crys}^*(M_n)$

, $M_n \in \underline{MF}_{tor}^{f, [0,r]}$

(forming proj. sys)

$\Rightarrow \frac{1}{p^{n_0}} \varprojlim M_n \longmapsto T$

($p^{n_0}T \supset p^{2n_0} T_{crys}^*(M) \subseteq D$

$\longrightarrow M_{n_0} \to p^{2n_0}M$ inj

strict so M_{n_0} & D induced fil

$\therefore$ str. div. lattice)

pf of Cor. C $M \in \underline{MF}^{f,[0,r]}_{tor}$

Using $M \supset M^i$ dir. summand,

$\exists$ resolution $0 \to L_1 \to L_2 \to M \to 0$

w/ $L_i \in \underline{MF}^{fd,[0,r]}_{w}$

Then $T_{cwys}^*(M) \simeq T_{cwys}^*(L_1) / T_{cwys}^*(L_2)$

& get $\begin{cases} \text{ess. surj by Thm B} \\ (\text{fully faithful by Thm A}) \end{cases}$

---

pf of Thm D : May assume $k = \bar{k}$

- Let E/K be alg ext, $m > 1 + \frac{r}{p-1}$ &

$\eta : \mathcal{O}_L \to \mathcal{O}_E / \mathfrak{a}^m_{E/K}$ $\mathcal{O}_K$-alg hom

- $P(X) :=$ the minimal pol of π_L / K

$$= X^{e_{L/K}} + \sum_{s=0}^{e_{L/K}-1} p c_s X^s$$

- $\hat{x} \in \mathcal{O}_E$: a lift of $\eta(\pi_L)$

Then :

- $P(\hat{x}) + \delta = 0$ with $\begin{cases} \delta \in \mathcal{O}_E \\ v_K(\delta) > 1 + \frac{r}{p-1} > 1 \end{cases}$

thus $v_K(\hat{x}) = \frac{1}{e_{L/K}} = v_K(\pi_L)$

"η preserves valuations" of both sides

- Hence

η induces $\bar{\eta} : \mathcal{O}_L / \mathfrak{b}_L \hookrightarrow \mathcal{O}_E / \mathfrak{b}_E$

: inj respecting fil

Claim $\bar{\eta}$ respects φ^i

pf $(\mathcal{O}_L / \mathfrak{b}_L)^i$ is generated$/k$ by

$\{ \pi_L^j \mid {}^{j \in \mathbb{Z}}_{j \geq \frac{e_{L/K} \cdot i}{p}} \}$

put $pj = e_{L/K} i + l$

Then :

$$\varphi^i(\pi_L^{j}) = \frac{\pi_L^{pj}}{(-p)^i} = \frac{\pi_L^{e_{L/K} i + l}}{(-p)^i}$$

$$= \pi_L^{l} \left(\sum_{s=0}^{e_{L/K}-1} C_s \pi_L^{s} \right)^i$$

$$\frac{\pi^{pj}}{(-p)^i}$$

$$\varphi^i(\eta(\pi_L)^j) = \pi^l \left(\sum_{s=0}^{e_{L/K}-1} C_s \pi^s + \frac{\delta}{p} \right)^i$$

& $\frac{\delta}{p} \in b_E$.

$\square$

Thus we get an inj

$$\mathrm{Tcrys}^*_{,L}(M) = \mathrm{Hom}_{\underline{MF}_{\tilde{k}}}(M, \mathcal{O}_L/b_L)$$

$$\downarrow$$

$$\mathrm{Tcrys}^*_{,E}(M) = \mathrm{Hom}_{\underline{MF}_{\tilde{k}}}(M, \mathcal{O}_E/b_E)$$

By key lemma 4. $\underline{L10}$

get $G_E \curvearrowright \mathrm{Tcrys}^*(M)$ trivial

i.e. $L \hookrightarrow E$ & (P_m) holds.

&

Fontaine - Yoshida's lemma

$$\Rightarrow \quad u_{L/K} \leq 1 + \frac{h}{p-1}$$

$\square$