Stratifying triangulated categories, I
Support and cosupport of complexes

Henning Krause

Universität Bielefeld

Summer School – Derived Categories
Nantes
June 23, 2014

www.math.uni-bielefeld/~hkrause

An outline

Support and cosupport provide a link between

ALGEBRA AND GEOMETRY.

I discuss two papers of Amnon Neeman involving these concepts:

- *The chromatic tower of* $\mathsf{D}(R)$, Topology (1992).
- *Colocalizing subcategories of* $\mathsf{D}(R)$, Crelles Journal (2011).

Applications in representation theory of finite groups follow at the end.

All this is part of a joint project with D. Benson and S. Iyengar.

The setup

Here is the setup:

- $R =$ a commutative noetherian ring
- $\operatorname{Mod} R =$ the category of R-modules
- $\mathrm{D}(R) =$ the (unbounded) derived category of $\operatorname{Mod} R$
- $\operatorname{Spec} R =$ the set of prime ideals of R

$\mathrm{D}(R)$ is a triangulated category with set-indexed (co)products.

Localizing and colocalizing subcategories

Definition

A triangulated subcategory $\mathsf{C} \subseteq \mathsf{D}(R)$ is called

- localizing if C is closed under taking all coproducts,
- colocalizing if C is closed under taking all products.

For any class $\mathsf{S} \subseteq \mathsf{D}(R)$ write:

$$\begin{aligned} \operatorname{Loc}(\mathsf{S}) &= \text{ the smallest localizing subcategory containing } \mathsf{S} \\ \operatorname{Coloc}(\mathsf{S}) &= \text{ the smallest colocalizing subcategory containing } \mathsf{S} \end{aligned}$$

Classifying localizing subcategories

Theorem (Neeman, 1992)

The assignment

$$\operatorname{Spec} R \supseteq \mathcal{U} \longmapsto \operatorname{Loc}(\{k(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{U}\}) \subseteq \mathsf{D}(R)$$

induces a bijection between

- *the collection of subsets of* $\operatorname{Spec} R$*, and*
- *the collection of localizing subcategories of* $\mathsf{D}(R)$*.*

Notation: $k(\mathfrak{p})$ = the residue field $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$

Classifying colocalizing subcategories

Theorem (Neeman, 2011)

The assignment

$$\operatorname{Spec} R \supseteq \mathcal{U} \longmapsto \mathsf{Coloc}(\{k(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{U}\}) \subseteq \mathsf{D}(R)$$

induces a bijection between

- *the collection of subsets of* $\operatorname{Spec} R$, *and*
- *the collection of colocalizing subcategories of* $\mathsf{D}(R)$.

- This is surprising because products tend to be complicated!
- How are the results from '92 and '11 related to each other?
- Is there a common proof?

A consequence / reformulation

For $\mathsf{C} \subseteq \mathsf{D}(R)$ write:

$$\mathsf{C}^{\perp} = \{X \in \mathsf{D}(R) \mid \operatorname{Hom}_{\mathsf{D}(R)}(C, X) = 0 \text{ for all } C \in \mathsf{C}\}$$
$$^{\perp}\mathsf{C} = \{X \in \mathsf{D}(R) \mid \operatorname{Hom}_{\mathsf{D}(R)}(X, C) = 0 \text{ for all } C \in \mathsf{C}\}$$

- If C is localizing, then $\mathsf{C}^{\perp}$ is colocalizing.
- If C is colocalizing, then $^{\perp}\mathsf{C}$ is localizing.
- If C is localizing, then $^{\perp}(\mathsf{C}^{\perp}) = \mathsf{C}$ [Neeman 1992].

Corollary (Neeman, 2011)

The assignment $\mathsf{C} \mapsto \mathsf{C}^{\perp}$ induces a bijection between

- *the collection of localizing subcategories of $\mathsf{D}(R)$, and*
- *the collection of colocalizing subcategories of $\mathsf{D}(R)$.*

The support of a complex

Definition (Foxby, 1979)

For $X \in \mathrm{D}(R)$ define the support

$$\operatorname{supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X \otimes^{\mathbf{L}}_R k(\mathfrak{p}) \neq 0\}.$$

Some examples:

- If $X \in \mathrm{D}^b(\operatorname{mod} R)$, then

$$\operatorname{supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \neq 0\} = \bigcup_{n \in \mathbb{Z}} \operatorname{supp} H^n(X).$$

- Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $\operatorname{supp} E(R/\mathfrak{p}) = \operatorname{supp} k(\mathfrak{p}) = \{\mathfrak{p}\}$.

Corollary (Neeman, 1992)

For $X, Y \in \mathrm{D}(R)$ we have

$$\operatorname{supp} X \subseteq \operatorname{supp} Y \iff \operatorname{Loc}(X) \subseteq \operatorname{Loc}(Y).$$

A test question

Let us test our understanding of localizing subcategories:

Question

Let $X \in \mathsf{D}(R)$. Is it true that

$$\mathsf{Loc}(X) = \mathsf{Loc}(H^*(X))\,?$$

An example

$R = k[\![x,y]\!]$ (k a field)
$\mathfrak{m} = (x,y)$ the maximal ideal of R
$Q =$ the field of fractions of R

The minimal injective resolution of R:

$$\cdots \longrightarrow 0 \longrightarrow Q \longrightarrow \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} E(R/\mathfrak{p}) \longrightarrow E(R/\mathfrak{m}) \longrightarrow 0 \longrightarrow \cdots$$

$$X\colon \qquad \cdots \longrightarrow 0 \longrightarrow Q \longrightarrow \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} E(R/\mathfrak{p}) \longrightarrow 0 \longrightarrow \cdots$$

In $\mathrm{D}(R)$ one has

$$\operatorname{supp} X = (\operatorname{Spec} R) \setminus \{\mathfrak{m}\} \qquad \operatorname{supp} H^*(X) = \operatorname{Spec} R$$

Thus $\operatorname{Loc}(X) \neq \operatorname{Loc}(H^* X)$.

The cosupport of a complex

Definition

For $X \in \mathrm{D}(R)$ define the cosupport

$$\operatorname{cosupp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathbf{R}\mathrm{Hom}_R(k(\mathfrak{p}), X) \neq 0\}.$$

This seems hard to compute, even for 'simple' objects:

- Let $R = \mathbb{Z}$. Then $\operatorname{cosupp} X = \operatorname{supp} X$ for $X \in \mathrm{D}^b(\operatorname{mod} R)$.
- Let $(R, \mathfrak{m})$ be complete local. Then $\operatorname{cosupp} R = \{\mathfrak{m}\}$.

Proposition

For a complex X in $\mathrm{D}(R)$ we have

$$\operatorname{Max}(\operatorname{supp} X) = \operatorname{Max}(\operatorname{cosupp} X).$$

Notation: $\operatorname{Max} \mathcal{U} = \{\mathfrak{p} \in \mathcal{U} \mid \mathfrak{p} \subseteq \mathfrak{q} \in \mathcal{U} \implies \mathfrak{p} = \mathfrak{q}\}$.

Four fundamental functors

Four fundamental (idempotent) functors $\operatorname{Mod} R \to \operatorname{Mod} R$:

- localization $\quad M \longrightarrow M \otimes_R R_{\mathfrak{p}}$
- colocalization $\quad \operatorname{Hom}_R(R_{\mathfrak{p}}, M) \longrightarrow M$
- torsion $\quad \varGamma_{\mathfrak{a}} M = \varinjlim \operatorname{Hom}(R/\mathfrak{a}^n, M) \longrightarrow M$
- completion $\quad M \longrightarrow \varLambda_{\mathfrak{a}} M = \varprojlim M \otimes_R R/\mathfrak{a}^n$

Their derived functors $\mathrm{D}(R) \to \mathrm{D}(R)$:

- localization $\quad X \longrightarrow X \otimes_R^{\mathbf{L}} R_{\mathfrak{p}}$
- colocalization $\quad \mathbf{R}\operatorname{Hom}_R(R_{\mathfrak{p}}, X) \longrightarrow X$
- local cohomology $\quad \mathbf{R}\varGamma_{\mathfrak{a}} X \longrightarrow X$ [Grothendieck, 1967]
- local homology $\quad X \longrightarrow \mathbf{L}\varLambda_{\mathfrak{a}} X$ [Greenlees–May, 1992]

Note:

- The functor $\mathbf{R}\operatorname{Hom}_R(R_{\mathfrak{p}}, -)$ is a right adjoint of $- \otimes_R^{\mathbf{L}} R_{\mathfrak{p}}$.
- The functor $\mathbf{L}\varLambda_{\mathfrak{a}}$ is a right adjoint of $\mathbf{R}\varGamma_{\mathfrak{a}}$.

Local (co)homology

Definition

Fix $\mathfrak{p} \in \operatorname{Spec} R$ and define (by abuse of notation):

- local cohomology $\mathit{\Gamma}_{\mathfrak{p}} = \mathbf{R}\mathit{\Gamma}_{\mathfrak{p}}(- \otimes^{\mathbf{L}}_{R} R_{\mathfrak{p}})$,
- local homology $\mathit{\Lambda}_{\mathfrak{p}} = \mathbf{R}\mathrm{Hom}_{R}(R_{\mathfrak{p}}, \mathbf{L}\mathit{\Lambda}_{\mathfrak{p}}-)$.

These are idempotent functors $\mathrm{D}(R) \to \mathrm{D}(R)$, and $\mathit{\Lambda}_{\mathfrak{p}}$ is a right adjoint of $\mathit{\Gamma}_{\mathfrak{p}}$.

We consider their essential images:

- $\operatorname{Im} \mathit{\Gamma}_{\mathfrak{p}} =$ local cohomology objects (a localizing subcategory)
- $\operatorname{Im} \mathit{\Lambda}_{\mathfrak{p}} =$ local homology objects (a colocalizing subcategory)

Note: $\mathit{\Lambda}_{\mathfrak{p}}$ induces an equivalence $\operatorname{Im} \mathit{\Gamma}_{\mathfrak{p}} \xrightarrow{\sim} \operatorname{Im} \mathit{\Lambda}_{\mathfrak{p}}$.

(Co)support revisited

An alternative description of (co)support:

- $\operatorname{supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathit{\Gamma}_{\mathfrak{p}} X \neq 0\}$.
- $\operatorname{cosupp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathit{\Lambda}_{\mathfrak{p}} X \neq 0\}$.

The following are equivalent:

- $H^n(X)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion for all $n \in \mathbb{Z}$.
- $\operatorname{supp} X \subseteq \{\mathfrak{p}\}$.
- X lies in $\operatorname{Im} \mathit{\Gamma}_{\mathfrak{p}}$.

There seems to be no analogue for $\mathit{\Lambda}_{\mathfrak{p}}$.

Stratification of $\mathsf{D}(R)$

Proposition

The assignment

$$\mathsf{D}(R) \supseteq \mathsf{C} \longmapsto (\mathsf{C} \cap \operatorname{Im} \varGamma_{\mathfrak{p}})_{\mathfrak{p} \in \operatorname{Spec} R}$$

induces a bijection between

- *the collection of localizing subcategories of* $\mathsf{D}(R)$*, and*
- *the collection of families* $(\mathsf{C}_{\mathfrak{p}})_{\mathfrak{p} \in \operatorname{Spec} R}$ *with each* $\mathsf{C}_{\mathfrak{p}} \subseteq \operatorname{Im} \varGamma_{\mathfrak{p}}$ *a localizing subcategory.*

Analogously, the assignment

$$\mathsf{D}(R) \supseteq \mathsf{C} \longmapsto (\mathsf{C} \cap \operatorname{Im} \varLambda_{\mathfrak{p}})_{\mathfrak{p} \in \operatorname{Spec} R}$$

classifies the colocalizing subcategories of $\mathsf{D}(R)$.

(Co)localizing subcategories of $\mathsf{D}(R)$

Proposition

Let $\mathfrak{p} \in \operatorname{Spec} R$.

- $\operatorname{Im} \mathit{\Gamma}_{\mathfrak{p}}$ *has no proper localizing subcategories.*
- $\operatorname{Im} \mathit{\Lambda}_{\mathfrak{p}}$ *has no proper colocalizing subcategories.*

Proof.

For each $0 \neq X \in \operatorname{Im} \mathit{\Gamma}_{\mathfrak{p}}$, one shows that

$$\operatorname{Loc}(X) = \operatorname{Loc}(k(\mathfrak{p})) = \operatorname{Im} \mathit{\Gamma}_{\mathfrak{p}}.$$

Analogously, $\operatorname{Coloc}(Y) = \operatorname{Im} \mathit{\Lambda}_{\mathfrak{p}}$ for each $0 \neq Y \in \operatorname{Im} \mathit{\Lambda}_{\mathfrak{p}}$. □

The classifications of [Neeman, 1992] and [Neeman, 2011] are immediate consequences.

Formulas for support and cosupport

Theorem

For $X, Y \in \mathrm{D}(R)$ we have:

$$\operatorname{supp}(X \otimes^{\mathbf{L}}_R Y) = (\operatorname{supp} X) \cap (\operatorname{supp} Y)$$

$$\operatorname{cosupp}(\mathbf{R}\mathrm{Hom}_R(X, Y)) = (\operatorname{supp} X) \cap (\operatorname{cosupp} Y)$$

A generalization and an application

The above proof allows to generalize Neeman's results to the derived category of a differential graded algebra A such that

- A is formal, i.e. quasi-isomorphic to its cohomology $H^*(A)$,
- $H^*(A)$ is graded-commutative and noetherian.

An application to the study of modular representations of finite groups goes as follows:

Let G be a finite group and k a field of characteristic $p > 0$. We consider modules over the group algebra kG and classify the (co)localizing subcategories of the stable category $\mathrm{StMod}\, kG$.

Modular representations of finite groups

Take as example $G = (\mathbb{Z}/2\mathbb{Z})^r$ and a field k of characteristic 2.

Group algebra $kG \cong k[x_1, \ldots, x_r]/(x_1^2, \ldots, x_r^2)$
Group cohomology $H^*(G, k) = \operatorname{Ext}^*_{kG}(k, k) \cong k[\xi_1, \ldots, \xi_r]$

$\mathsf{K}(\operatorname{Inj} kG)$ = category of complexes of injective kG-modules / htpy.
$\mathbf{i}k$ = an injective resolution of the trivial representation k
$\operatorname{End}_{kG}(\mathbf{i}k)$ = the endomorphism dg algebra of $\mathbf{i}k$ (is formal)

$$\operatorname{StMod} kG \xrightarrow{\sim} \mathsf{K}_{\mathrm{ac}}(\operatorname{Inj} kG) \hookrightarrow \mathsf{K}(\operatorname{Inj} kG)$$
$$\xrightarrow[\operatorname{Hom}_{kG}(\mathbf{i}k,-)]{\sim} \mathsf{D}(\operatorname{End}_{kG}(\mathbf{i}k)) \xrightarrow{\sim} \mathsf{D}(k[\xi_1, \ldots, \xi_r])$$

Corollary

There are canonical bijections between

- *(co)localizing subcategories of* $\operatorname{StMod} kG$*, and*
- *sets of graded non-maximal prime ideals of* $H^*(G, k)$.

Outlook

The next lecture will focus on infinite methods:

- Compactly generated triangulated categories
- Bousfield localization
- Brown representability

Then we explain the stratification of triangulated categories using:

- Local cohomology functors
- Support of objects
- Local-global principle

My collaborators

Dave Benson and Srikanth Iyengar
(at an Oberwolfach Seminar 2010)