Stratification of triangulated categories

Henning Krause

Notes by Pieter Belmans*

Contents

*Based on a lecture series at the Summer school on Derived categories, June 23rd to 27th 2014 in Nantes.

1 Day 1: Support and cosupport of complexes

This lecture used slides, hence the notes are only some random facts mentioned during the presentation but not present on the actual slideshow. The slideshow can be found on the summer school's website.

Remark 1. In Neeman's book the notation for orthogonals is orthogonal to the notation used elsewhere.

Remark 2. Foxby's support is the "small support": it is not specialisation-closed. Hence it is not the same as support of a module.

Remark 3. The formalism of (co)support using local (co)homology is a good way of avoiding the use of field objects.

Remark 4. The equivalence $\operatorname{StMod} kG \cong \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}/kG)$ is obtained from the *Tate resolution*: as the injectives and projectives in kG coincide, one glues together an injective and a projective resolution for an object in the stable module category.

2 Day 2: Infinite methods

2.1 Compact objects

Let $\mathcal{T}$ be a triangulated category, denote Σ its shift or suspension, and assume that $\mathcal{T}$ has set-indexed coproducts.

Definition 5. An object X of $\mathcal{T}$ is *compact* if $\operatorname{Hom}_{\mathcal{T}}(X,-)\colon \mathcal{T} \to \mathrm{Ab}$ preserves coproducts.

Lemma 6. An object X is compact if and only if for all $X \to \coprod_{i\in I} Y_i$ there exists a factorisation through $\coprod_{i\in I} Y_i$, where $I_0 \subseteq I$ is a finite subset.

Remark 7. These compact objects serve as building blocks for the category $\mathcal{T}$ and they constitute a thick subcategory $\mathcal{T}^c$.

Definition 8. The category $\mathcal{T}$ is *compactly generated* if $\mathcal{T} = \operatorname{Loc}(\mathcal{C})$, for some set of compact objects $\mathcal{C}$.

Proposition 9. For a set of compact objects $\mathcal{C} \subseteq \mathcal{T}^c$ the following are equivalent:

1. $\operatorname{Loc}(\mathcal{C}) = \mathcal{T}$;
2. for all objects X in $\mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n C, X) = 0$ for all objects $C \in \mathcal{C}$ and $n \in \mathbb{Z}$ we have $X = 0$.

Proof. From (1) to (2) is easy: let X be an object of $\mathcal{T}$ and consider

(1) $\quad {}^{\perp}X := \{V \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(\Sigma^n V, X) = 0, \forall n \in \mathbb{Z}\}$

which is a localising subcategory of $\mathcal{T}$. If $\mathcal{C} \subseteq {}^{\perp}X$ then $X = 0$.

From (2) to (1) we have to use Brown representability (see later), which depends on the compactness of the objects. The inclusion $\operatorname{Loc}(\mathcal{C}) \hookrightarrow \mathcal{T}$ has a right adjoint Γ. Let X be an object of $\mathcal{T}$, by the adjunction we have a morphism $\Gamma(X) \to X$ that we can complete to a triangle

(2) $\quad \Gamma(X) \to X \to X' \to$

and the long exact sequence that we can obtain by applying $\mathrm{Hom}_{\mathcal{T}}(V,-)$ tells us that $\mathrm{Hom}_{\mathcal{T}}(V,X')=0$ for all $X\in\mathrm{Loc}(\mathcal{C})$. So if (2) holds then we have $X'=0$ and therefore $X\in\mathrm{Loc}(\mathcal{C})$. □

Proposition 10. Let X and Y be compact objects of $\mathcal{T}$. If $Y\in\mathrm{Loc}(X)$ then we already have $Y\in\mathrm{Thick}(X)$.

Hence if a compact object Y can be obtained from a compact object X in infinitely many steps it can also be done in finitely many.

Example 11. Let A be any ring and take $X\in\mathbf{D}(\mathrm{Mod}/A)$. Then the following are equivalent:

1. $X\in\mathrm{Thick}(A)$;
2. X is isomorphic to a perfect complex;
3. X is compact.

2.2 Brown representability

Theorem 12. Let $\mathcal{T}$ be a compactly generated triangulated category. For a functor $H\colon\mathcal{T}^{\mathrm{op}}\to\mathrm{Ab}$ the following are equivalent:

1. H is cohomological, i.e. it sends exact triangles in $\mathcal{T}$ to exact sequences and it sends coproducts in $\mathcal{T}$ to products;
2. H is representable, i.e. there exists an object $X\in\mathcal{T}$ such that $H\cong\mathrm{Hom}_{\mathcal{T}}(-,X)$.

Corollary 13. Let $\mathcal{T}$ be a compactly generated triangulated category. Then $\mathcal{T}$ has all coproducts.

Proof. Consider $\{X_i\}_{i\in I}$ a family of objects in $\mathcal{T}$. Then $\prod_{i\in I}\mathrm{Hom}_{\mathcal{T}}(-,X_i)$ satisfies condition (1) in the Brown representability theorem, hence it must be representable by an object that satisfies the universal property for a product. □

Corollary 14. Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{U}$ be any triangulated category. Then for all exact functors $F\colon\mathcal{T}\to\mathcal{U}$ the following are equivalent:

1. F preserves coproducts;
2. F admits a right adjoint G.

Proof. From (1) to (2) we consider any object X in $\mathcal{U}$, then the functor

$$\mathrm{Hom}_{\mathcal{U}}(F(-),X)\colon\mathcal{T}^{\mathrm{op}}\to\mathrm{Ab} \tag{3}$$

satisfies condition (1) in the Brown representability theorem, hence it can be represented as $\mathrm{Hom}_{\mathcal{T}}(-,Y)$, and set $G(X):=Y$.

From (2) to (1) is trivial as left adjoints preserve coproducts. □

We will often take $\mathcal{U}$ a Verdier quotient in applications.

2.3 Bousfield localisation

Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a thick subcategory. Its *Verdier quotient*

(4) $\quad \mathcal{T}/\mathcal{S} := \mathcal{T}[\{\sigma \in \operatorname{Mor}(\mathcal{T}) \mid \operatorname{cone}(\sigma) \in \mathcal{S}\}^{-1}]$

is again a triangulated category, the quotient functor $Q\colon \mathcal{T} \to \mathcal{S}$ is exact and its kernel $\ker(Q) = \mathcal{S}$, where the *kernel* and *essential image*

(5)
$$\begin{aligned} \ker(F) &:= \{X \in \mathcal{T} \mid F(X) = 0\} \\ \operatorname{im}(F) &:= \{Y \in \mathcal{U} \mid \exists X \in \mathcal{T}\colon Y \cong F(X)\} \end{aligned}$$

are full triangulated subcategories.

Example 15. Let $\mathcal{A}$ be an abelian category. Then $\mathbf{D}(\mathcal{A}) := \mathbf{K}(\mathcal{A})/\mathbf{K}_{\mathrm{ac}}(\mathcal{A})$ is a triangulated category.

Remark 16. A priori it is unclear whether a Verdier quotient $\mathcal{T}/\mathcal{S}$ is locally small.

Definition 17. A *localisation functor* is an exact endofunctor $L\colon \mathcal{T} \to \mathcal{T}$ if there is a natural transformation $\eta\colon \mathrm{id}_{\mathcal{T}} \Rightarrow L$ such that

1. $L \circ \eta\colon L \Rightarrow L^2$ is invertible, i.e. for all X the morphism $L(\eta_X)\colon L(X) \to L^2(X)$ is an isomorphism;
2. $L \circ \eta = \eta \circ L$, i.e. $L(\eta_X) = \eta_{L(X)}$.

Definition 18. A *colocalisation functor* is an exact endofunctor $\Gamma\colon \mathcal{T} \to \mathcal{T}$ for which the opposite functor Γ^{op} is a localisation functor.

Remark 19. The notation Γ is a reference both to Grothendieck's local cohomology functor and the symmetry between L and Γ.

Proposition 20. For a thick subcategory $\mathcal{S}$ of $\mathcal{T}$ the following are equivalent:

1. $\mathcal{S} \hookrightarrow \mathcal{T}$ admits a right adjoint;
2. $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ admits a right adjoint;
3. there exists a localisation functor L such that $\ker(L) = \mathcal{S}$;
4. there exists a colocalisation functor Γ such that $\operatorname{im}(\Gamma) = \mathcal{S}$.

In this case, the following holds:

1. for all objects X in $\mathcal{T}$ there exists a functorial triangle

   (6) $\quad \Gamma(X) \to X \to L(X) \to$

2. $\mathcal{S}^{\perp} = \operatorname{im}(L) = \ker(\Gamma)$ and ${}^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$ (where a priori the left-hand side could be bigger);
3. Γ induces a right adjoint for the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$;
4. L induces a left adjoint for the inclusion $\mathcal{S}^{\perp} \hookrightarrow \mathcal{T}$;

Remark 21. This is related to the notion of semi-orthogonal decompositions: we obtain $\mathcal{T} = \langle \mathcal{S}^{\perp}, \mathcal{S} \rangle = \langle \mathcal{S}, {}^{\perp}\mathcal{S} \rangle$. A useful mnemonic (suggested by Sasha Kuznetsov) is that $\perp$ is always in the middle.

Example 22. Let $\mathcal{T}$ be a compactly generated triangulated category. Let $\mathcal{S} = \operatorname{Loc}(\mathcal{C})$ be a localising subcategory generated by a set of compact objects $\mathcal{C}$. We obtain a diagram

$$\begin{array}{ccccc} \mathcal{S}^c & \hookrightarrow & \mathcal{T}^c & \twoheadrightarrow & \mathcal{T}^c/\mathcal{S}^c \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S} & \hookrightarrow & \mathcal{T} & \twoheadrightarrow & \mathcal{T}/\mathcal{S} \end{array} \tag{7}$$

where $\mathcal{S}^c = \operatorname{Thick}(\mathcal{C})$ and the induced functor $\mathcal{T}^c/\mathcal{S}^c \to (\mathcal{T}/\mathcal{S})^c$ is an equivalence up to direct summands.

2.4 Describing localising subcategories

The problem that we would like to solve is the description of all localising subcategories of a compactly generated triangulated category. Some obvious questions that arise are:

1. do they form a set or a proper class?
2. how can we explicitly describe them?

Definition 23. Let $\mathcal{T}$ be a triangulated category that has coproducts. Then $\mathcal{T}$ is *well-generated* if $\mathcal{T} = \operatorname{Loc}(\mathcal{C})$, where $\mathcal{C}$ is a set of α-compact objects, and α is a regular cardinal.

The case $\alpha = \aleph_0$ yields the usual definition of a compactly generated category.

Theorem 24. Brown representability holds for well-generated categories.

Corollary 25. Let $\mathcal{T}$ be a compactly generated triangulated category. Let $\mathcal{C}$ be any set of objects of $\mathcal{T}$ (we no longer assume the objects to be compact). Then $\operatorname{Loc}(\mathcal{C}) \hookrightarrow \mathcal{T}$ admits a right adjoint.

Exercise 26. The corollary is due to Neeman; find this important result in his book on triangulated categories (it's well hidden, unfortunately).

Theorem 27. Let $\mathcal{T}$ be a compactly generated triangulated category. Then

$$\operatorname{card}\{\operatorname{Loc}(\mathcal{C}) \subseteq \mathcal{T} \mid \mathcal{C} \text{ a set}\} \leq 2^{2^{\#\mathcal{T}^c}} \tag{8}$$

where $\#\mathcal{T}^c := \operatorname{card}(\operatorname{Mor}(\mathcal{T}^c))$.

3 Day 3: Stratification of big triangulated categories

3.1 Rings acting on triangulated categories

Recall that for a commutative noetherian ring A we have the associated (big) derived category $\mathbf{D}(A) := \mathbf{D}(\operatorname{Mod}/A)$, and to an object X in $\mathbf{D}(A)$ we have associated a support $\operatorname{supp} X$ and cosupport $\operatorname{cosupp} X$ which are subsets of $\operatorname{Spec} A$. This way we parametrise localising and colocalising subcategories. More precisely, the main results from the first lecture are:

Theorem 28. Let X and Y be objects of $\mathbf{D}(A)$. Then $\operatorname{supp} X \subseteq \operatorname{supp} Y$ if and only if $\operatorname{Loc}(X) \subseteq \operatorname{Loc}(Y)$.

Theorem 29. Let X and Y be objects of $\mathbf{D}(A)$. Then $\operatorname{cosupp} X \subseteq \operatorname{cosupp} Y$ if and only if $\operatorname{Coloc}(X) \subseteq \operatorname{Coloc}(Y)$.

Corollary 30. Let X and Y be objects of $\mathbf{D}(A)$. Then $\operatorname{Ext}^{\bullet}_A(X,Y) = 0$ if and only if $\operatorname{supp} X \cap \operatorname{cosupp} Y = \emptyset$.

Now more generally, let $\mathcal{T}$ be a triangulated category.

Problem How can we determine that for X and Y objects in $\mathcal{T}$ we have

$$\operatorname{Hom}^{\bullet}_{\mathcal{T}}(X,Y) := \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(X,\Sigma^n Y) = 0? \tag{9}$$

Fix a $\mathbb{Z}$-graded commutative ring $R = \bigoplus_{n\in\mathbb{Z}} R^n$, i.e. we have $rs = (-1)^{|r||s|}sr$ for r and s homogeneous elements. This graded-commutativity enters the picture because we will often take R to be a cohomology ring.

Definition 31. Let $\mathcal{T}$ be a triangulated category and R a graded commutative ring. We say that $\mathcal{T}$ is *R-linear*, or that R *acts on* $\mathcal{T}$ if there is a homomorphism of rings

$$\varphi\colon R \to \mathrm{Z}^{\bullet}(\mathcal{T}) \tag{10}$$

where $\mathrm{Z}^{\bullet}(\mathcal{T})$ is the *graded centre* of $\mathcal{T}$, which is the graded commutative ring whose degree n piece is given by

$$\mathrm{Z}^n(\mathcal{T}) := \left\{\eta\colon \mathrm{id}_{\mathcal{T}} \Rightarrow \Sigma^n \mid \eta\circ\Sigma = (-1)^n \Sigma\circ\eta\right\}. \tag{11}$$

So for an object X of $\mathcal{T}$ we get a ring homomorphism

$$\varphi_X\colon R \to \operatorname{End}^{\bullet}_{\mathcal{T}}(X) \tag{12}$$

and $\operatorname{Hom}^{\bullet}_{\mathcal{T}}(X,Y)$ is a graded R-module via φ_X acting on the right and φ_Y acting on the left, whose actions coincide up to a sign.

Standing assumptions From now on we take R a graded commutative noetherian ring, $\mathcal{T}$ a compactly generated triangulated category and $\operatorname{Spec} R$ the set of graded prime ideals of R (which might be confusing at first for algebraic geometers).

To a graded ideal $\mathfrak{a}$ we assign

$$\mathrm{V}(\mathfrak{a}) := \left\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\right\} \tag{13}$$

and an R-module M is $\mathfrak{a}$-*torsion* if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \mathrm{V}(\mathfrak{a})$, which is equivalent to the usual definition of torsion.

A subset V is *specialisation-closed* if for all $\mathfrak{p} \subseteq \mathfrak{q}$ such that $\mathfrak{p} \in V$ we have $\mathfrak{q} \in V$.

Definition 32. An object X of $\mathcal{T}$ is V-*torsion* if $\operatorname{Hom}^{\bullet}_{\mathcal{T}}(C,X)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R \setminus V$ and all $C \in \mathcal{T}^{\mathrm{c}}$.

We then set

$$\mathcal{T}_V := \{X \in \mathcal{T} \mid X \text{ is } V\text{-torsion}\} \tag{14}$$

which is a localising subcategory as the vanishing condition uses compact objects.

Proposition 33. For a specialisation-closed subset V there are

1. a localisation functor $\mathrm{L}_V : \mathcal{T} \to \mathcal{T}$ such that $\ker \mathrm{L}_V = \mathcal{T}_V$,
2. a colocalisation functor $\Gamma_V : \mathcal{T} \to \mathcal{T}$ such that $\operatorname{im} \Gamma_V = \mathcal{T}_V$,

such that we obtain a *localisation triangle*

(15) $\Gamma_V(X) \to X \to \mathrm{L}_V(X) \to$

for all objects X in $\mathcal{T}$.

For $\mathfrak{p} \in \operatorname{Spec} R$ we set the *localisation* of an object X of $\mathcal{T}$ at $\mathfrak{p}$ to be

(16) $X_{\mathfrak{p}} := \mathrm{L}_{\mathrm{Z}(\mathfrak{p})}(X)$

where $\mathrm{Z}(\mathfrak{p}) := \{\mathfrak{q} \mid \mathfrak{q} \nsubseteq \mathfrak{p}\} = \operatorname{Spec} R \setminus \operatorname{Spec} R_{\mathfrak{p}}$. The natural map $X \to X_{\mathfrak{p}}$ obtained from the adjunction induces an isomorphism

(17) $\operatorname{Hom}^\bullet_{\mathcal{T}}(C, X)_{\mathfrak{p}} \cong \operatorname{Hom}^\bullet_{\mathcal{T}}(C, X_{\mathfrak{p}})$

for all compact objects C in $\mathcal{T}$.

Definition 34. We say that an object X of $\mathcal{T}$ is $\mathfrak{p}$-*local* if $X \cong X_{\mathfrak{p}}$.

So now we have the notion of $\mathfrak{p}$-local objects and $\mathfrak{p}$-torsion objects.

Definition 35. For $\mathfrak{p} \in \operatorname{Spec} R$ we set the *local cohomology* of an object X to be

(18) $\Gamma_{\mathfrak{p}}(X) := \Gamma_{\mathrm{V}(\mathfrak{p}} \circ \mathrm{L}_{\mathrm{Z}(\mathfrak{p})}(X) = \Gamma_{\mathrm{V}(\mathfrak{p})}(X_{\mathfrak{p}})$.

This is an idempotent functor, and $X \in \Gamma_{\mathfrak{p}}(\mathcal{T})$ if and only if X is both $\mathfrak{p}$-local and $\mathfrak{p}$-torsion.

Remark 36. The notation is not a coincidence: it coincides with Grothendieck's local cohomology functor in the appropriate setting.

Lemma 37. Let V and W be specialisation-closed subsets of $\operatorname{Spec} R$ such that we have $V \setminus W = \{\mathfrak{p}\}$. Then

(19) $\Gamma_V \circ \mathrm{L}_W \cong \Gamma_{\mathfrak{p}} \cong \mathrm{L}_W \circ \Gamma_V$.

This can for instance be applied to $V = \mathrm{V}(\mathfrak{p})$ and $W = \mathrm{Z}(\mathfrak{p})$.

3.2 Stratification of big triangulated categories

We now have the necessary tools to introduce stratification.

Definition 38. Let X be an object of $\mathcal{T}$, with R acting on $\mathcal{T}$. Then

(20) $\operatorname{supp}_R X := \left\{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_{\mathfrak{p}}(X) \neq 0\right\}$

is the *support* of X.

Remark 39. If $X \to Y \to Z \to$ is an exact triangle, then

(21) $\operatorname{supp}_R Y \subseteq \operatorname{supp}_R X \cup \operatorname{supp}_R Y$

and

(22) $\operatorname{supp}_R \left(\coprod_{i \in I} X_i \right) = \bigcup_{i \in I} \operatorname{supp}_R X_i$.

Theorem 40. For an object X of $\mathcal{T}$ we have

$$\operatorname{supp}_R X = \bigcup_{C \in \mathcal{T}^c} \min\{\operatorname{Supp}_R \operatorname{Hom}^\bullet_{\mathcal{T}}(C, X)\} \tag{23}$$

where for an R-module M

$$\operatorname{Supp}_R M := \left\{ \mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \neq 0 \right\} \tag{24}$$

and a subset $\mathcal{U} \subseteq \operatorname{Spec} R$

$$\min \mathcal{U} := \{\mathfrak{p} \in \mathcal{U} \mid \mathfrak{q} \in \mathcal{U}, \mathfrak{q} \subseteq \mathfrak{p} \Rightarrow \mathfrak{q} = \mathfrak{p}\}\,. \tag{25}$$

Hence the minimal elements determine the support: as we are working with specialisation-closed subsets, the support supp_R only considers the minimal primes.

As we have assumed $\mathcal{T}$ to be compactly generated we get the following corollary.

Corollary 41. For all objects X in $\mathcal{T}$ we have $\operatorname{supp}_R X = \emptyset$ if and only if $X = 0$.

We now come to the main definition of the lecture series.

Definition 42. The category $\mathcal{T}$ is *stratified by (the action of) R* if

1. we have that

   $$\operatorname{Loc}(X) = \operatorname{Loc}(\{\Gamma_\mathfrak{p}(X) \mid \mathfrak{p} \in \operatorname{Spec} R\}) \tag{26}$$

   for all objects X in $\mathcal{T}$, i.e. we have the *local-to-global principle* which says that objects are built up from $\mathfrak{p}$-local and $\mathfrak{p}$-torsion information;
2. $\Gamma_\mathfrak{p}(\mathcal{T})$ has no proper localising subcategories, for all $\mathfrak{p} \in \operatorname{Spec} R$.

Proposition 43. Let $\mathcal{T}$ be as above.

1. The local-to-global principle holds whenever the Krull dimension of R is finite or when $\mathcal{T}$ "has a model."
2. Suppose that the local-to-global-principle holds. Then we have bijections

   $$\begin{array}{ccc} \{\text{localising subcategories of } \mathcal{T}\} & & \mathcal{S} \subseteq \mathcal{T} \\ \updownarrow \scriptstyle{1:1} & & \downarrow \\ \left\{\begin{array}{c}\text{collections of localising subcategories of } \mathcal{T}_\mathfrak{p} \\ \text{for all } \mathfrak{p} \in \operatorname{Spec} R\end{array}\right\} & & (\mathcal{S} \cap \Gamma_\mathfrak{p}(\mathcal{T}))_{\mathfrak{p} \in \operatorname{Spec} R}. \end{array} \tag{27}$$

Example 44.

1. Neeman: $\mathbf{D}(A)$ is stratified by $R = A$, where A is a commutative noetherian ring.
2. Benson–Iyengar–Krause: $\operatorname{StMod}(kG)$ is stratified by $\mathrm{H}^\bullet(G, k)$ (where G is a finite p-group).

3.3 Consequences of stratification

Theorem 45. Suppose that $\mathcal{T}$ is stratified by R. Then we have a bijection

$$(28)\qquad \begin{array}{ccc} \{\text{localising subcategories of } \mathcal{T}\} & & \mathcal{S} \subseteq \mathcal{T} \\ \updownarrow \scriptstyle{1:1} & & \downarrow \\ \{\text{subsets of } \operatorname{supp}_R(\mathcal{T})\} & & \operatorname{supp}_R(\mathcal{S}) := \bigcup_{X\in\mathcal{S}} \operatorname{supp}_R X. \end{array}$$

We say $\mathcal{T}$ is *noetherian* if $\operatorname{Hom}^{\bullet}_{\mathcal{T}}(X,Y)$ is finitely generated over R for all compact objects X and Y of $\mathcal{T}$.

Lemma 46. Let X be an object of $\mathcal{T}$. If $\operatorname{End}^{\bullet}_{\mathcal{T}}(X)$ is finitely generated over R then $\operatorname{supp}_R X = \mathrm{V}(\mathfrak{a})$, for $\mathfrak{a} := \ker(R \to \operatorname{End}^{\bullet}_{\mathcal{T}}(X))$.

This allows us to classify thick subcategories of $\mathcal{T}^{\mathrm{c}}$!

Theorem 47. Let $\mathcal{T}$ be as above, and assume moreover that $\mathcal{T}$ is noetherian. Then we have a bijection

$$(29)\qquad \begin{array}{ccc} \{\text{thick subcategories of } \mathcal{T}^{\mathrm{c}}\} & & \mathcal{S} \subseteq \mathcal{T}^{\mathrm{c}} \\ \updownarrow \scriptstyle{1:1} & & \downarrow \\ \{\text{specialisation-closed subsets of } \operatorname{supp}_R(\mathcal{T})\} & & \operatorname{supp}_R(\mathcal{S}). \end{array}$$

Corollary 48. Let $\mathcal{T}$ be as above, and assume moreover that $\mathcal{T}$ is noetherian. Then for all compact objects X and Y of $\mathcal{T}$ we have

$$(30)\qquad \operatorname{Supp}_R \operatorname{Hom}^{\bullet}_{\mathcal{T}}(X,Y) = \operatorname{supp}_R X \cap \operatorname{supp}_R Y.$$

Example 49.
1. If $(\mathcal{T},\otimes,\mathbf{1})$ is a tensor triangulated category then $\mathcal{T}$ is $\operatorname{End}^{\bullet}_{\mathcal{T}}(\mathbf{1})$-linear, and we obtain a classification of tensor ideal localising subcategories. This generalises the classification of localising subcategories to any finite group G, without the assumption on the order of G.
2. Let A be a finite-dimensional k-algebra. If we wish to stratify $\mathbf{D}(A)$ we may run into the following problems:
   (a) The category of compacts is *too small*. Take $A = kG$ for G a finite group, then $\mathbf{D}(A)^{\mathrm{c}} \cong \mathbf{D}^{\mathrm{b}}(\operatorname{proj}/A)$, but we'd rather study $\operatorname{stmod}(A)$;
   (b) The graded center is *too small*. Take $A = kQ$ for Q an acyclic quiver, then $\mathrm{Z}^{\bullet}(\mathbf{D}(A)) = k$ if Q is Dynkin, which is of no use to us.
3. On the other hand, $\mathrm{Z}^{\bullet}(\mathcal{T})$ can also be *too big* for meaningful calculations, in which case we will often take $\mathrm{HH}^{\bullet}(A/k)$ as the ring acting on $\mathcal{T}$, instead of the entire graded centre.

4 Day 4: $\mathbf{K}(\mathrm{Inj}/X)$ and Grothendieck duality

4.1 $\mathbf{D}^{\mathrm{b}}(X)$ and compact objects

Let X be a separated noetherian scheme. We have a chain of inclusions

$$\mathbf{D}^{\mathrm{perf}}(X) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X) \hookrightarrow \mathbf{D}(\mathrm{Qcoh}/X) \tag{31}$$

of triangulated categories. We can relate the outer two by the following proposition:

Proposition 50. The derived category $\mathbf{D}(\mathrm{Qcoh}/X)$ is compactly generated, with $\mathbf{D}(\mathrm{Qcoh}/X)^{\mathrm{c}} \cong \mathbf{D}^{\mathrm{perf}}(X)$.

But for Grothendieck duality (amongst other reasons) we are mostly interested in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$, which contains noncompact objects if X is not regular. Hence the category $\mathbf{D}(\mathrm{Qcoh}/X)$ is "too small" to have $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$ as its category of compacts.

To remedy this, observe that Qcoh/X is a Grothendieck category, hence it has enough injective objects, so we have $\mathrm{Inj}/X \hookrightarrow \mathrm{Qcoh}/X$, where Inj/X is closed under coproducts. Writing down the definitions for the derived category we get the following situation:

$$\begin{array}{ccccc}
\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}/X) & \hookrightarrow & \mathbf{K}(\mathrm{Inj}/X) & \twoheadrightarrow & \mathbf{K}(\mathrm{Inj}/X)/\,\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}/X) \\
\downarrow & & \downarrow & \searrow^{Q} & \downarrow \cong \\
\mathbf{K}_{\mathrm{ac}}(\mathrm{Qcoh}/X) & \hookrightarrow & \mathbf{K}(\mathrm{Qcoh}/X) & \twoheadrightarrow & \mathbf{D}(\mathrm{Qcoh}/X).
\end{array} \tag{32}$$

The following proposition shows that $\mathbf{K}(\mathrm{Inj}/X)$ is the "bigger category" that we need, with the desired compact objects.

Proposition 51. The category $\mathbf{K}(\mathrm{Inj}/X)$ is compactly generated, and Q induces an equivalence $\mathbf{K}(\mathrm{Inj}/X)^{\mathrm{c}} \cong \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$.

Proof. The set of compact generators corresponds to injective resolutions of coherent sheaves. Checking that these are compact boils down to studying coh/X inside Qcoh/X, and that they are generating is proved using Baer's injectivity criterion. □

Theorem 52. The functor Q admits a left and right adjoint. This yields a recollement

$$\mathbf{S}(\mathrm{Qcoh}/X) \hookrightarrow \mathbf{K}(\mathrm{Inj}/X) \twoheadrightarrow \mathbf{D}(\mathrm{Qcoh}/X) \tag{33}$$

where $\mathbf{S}(\mathrm{Qcoh}/X) \coloneqq \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}/X)$ is also known as the singularity category of X.

Proof. Brown representability gives us the right adjoint for Q, because Inj/X is closed under coproducts.

The proof for the left adjoint went a bit haywire and is not reproduced here. □

There is a remarkable consequence, given that taking products in Qcoh/X in general might not be exact.

Corollary 53. A product of acyclic complexes of injectives is again acyclic.

Remark 54. Denoting the left and right adjoint by Q_λ and Q_ρ respectively, we get that

$$
\begin{aligned}
\operatorname{im} Q_\lambda &= {}^{\perp}\mathbf{S}(\mathrm{Qcoh}/X), \\
\operatorname{im} Q_\rho &= \mathbf{S}(\mathrm{Qcoh}/X)^{\perp} = \text{K-injective complexes.}
\end{aligned} \tag{34}
$$

Exercise 55. Based on a question by Lunts–Schnürer: the right adjoint Q_ρ identifies $\mathbf{D}(\mathrm{Qcoh}/X)$ with the full subcategory of K-injective complexes. Are they closed under taking coproducts? (Hint: if and only if X is regular)

The following lemma is useful for both the exercise and the subsequent corollary.

Lemma 56. Given an adjoint pair of exact functors between compactly generated triangulated categories, the left adjoint preserves compactness if and only if the right adjoint preserves coproducts.

Corollary 57. The upper row of the recollement yields (as left adjoints preserve compacts)

$$
\begin{array}{ccccc}
\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)/\mathbf{D}^{\mathrm{perf}}(X) & \twoheadleftarrow & \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X) & \hookleftarrow & \mathbf{D}^{\mathrm{perf}}(X) \\
\downarrow & & \downarrow \cong & & \downarrow \cong \\
\mathbf{S}(\mathrm{Qcoh}/X)^{\mathrm{c}} & \twoheadleftarrow & \mathbf{K}(\mathrm{Inj}/X)^{\mathrm{c}} & \hookleftarrow & \mathbf{D}(\mathrm{Qcoh}/X)^{\mathrm{c}}
\end{array} \tag{35}
$$

where the first vertical functor is an equivalence up to direct summands: $\mathbf{S}(\mathrm{Qcoh}/X)^{\mathrm{c}}$ is karoubian but the domain is not in general.

The category $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)/\mathbf{D}^{\mathrm{perf}}(X)$ occurs in two contexts:

1. singularity categories, by Orlov,
2. stable derived categories, by Buchweitz,

which explains the notation.

Remark 58. This works for any locally noetherian Grothendieck category, provided that $\mathbf{D}(\mathcal{A})$ is compactly generated.

Example 59. Take k a field of characteristic p and G a finite p-group (in the more general context of an arbitrary finite group one has to restrict to $\otimes$-ideals). Then we have a recollement

$$
\begin{array}{ccccc}
\operatorname{StMod} kG & \hookrightarrow & \mathbf{K}(\mathrm{Inj}/kG) & \twoheadrightarrow & \mathbf{D}(kG) \\
\\
\operatorname{Proj} \mathrm{H}^{\bullet}(G,k) & & \mathrm{H}^{\bullet}(G,k) & & *
\end{array} \tag{36}
$$

The spectra of the graded commutative rings in the bottom row describe the localising subcategories. We see that in this case $\mathbf{D}(kG)$ is a minimal (co)localising subcategory, hence corresponds to the unique maximal prime ideal. The category on the left is the one we are the most interested in, as it tells us the most about the representation theory of G.

Example 60. The category $\mathbf{S}(\mathrm{Qcoh}/X)$ has been stratified (via tensor action) when X is locally a complete intersection, this is a result by Greg Stevenson.

4.2 Grothendieck duality

Let A be a commutative noetherian ring. Associated to A we have

$$\begin{array}{ll} \mathrm{Inj}/A & \text{injective } A\text{-modules} \\ \mathrm{Proj}/A & \text{projective } A\text{-modules} \\ \mathrm{Flat}/A & \text{flat } A\text{-modules} \\ \mathbf{D}^{\mathrm{fin}}(A) & \left\{ X \in \mathbf{D}(\mathrm{Mod}/A) \mid \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^n(X) \text{ finitely generated over } A \right\} \end{array} \tag{37}$$

and we will be considering the situation

$$\begin{array}{ccccc} \mathrm{mod}/A & & \hookrightarrow & & \mathrm{Mod}/A \\ \downarrow & & & & \downarrow \\ \mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A) & \xrightarrow{\cong} & \mathbf{D}^{\mathrm{fin}}(A) & \hookrightarrow & \mathbf{D}(\mathrm{Mod}/A). \end{array} \tag{38}$$

We wish to find the "infinite completion" of Grothendieck duality. I.e. suppose that A admits a dualising complex D_A, which is a complex of injective objects such that

$$\mathbf{D}^{\mathrm{fin}}(A) \xrightarrow{\mathbf{R}\mathrm{Hom}_A(-,D_A)} \mathbf{D}^{\mathrm{fin}}(A) \tag{39}$$

is an equivalence. Can this be lifted to unbounded complexes?

One can show that there is a diagram

$$\mathbf{K}(\mathrm{Proj}/A) \rightleftarrows \mathbf{K}(\mathrm{Flat}/A) \underset{\mathrm{Hom}_A(D_A,-)}{\overset{-\otimes_A D_A}{\rightleftarrows}} \mathbf{K}(\mathrm{Inj}/A) \tag{40}$$

where the right adjoint to the inclusion $\mathbf{K}(\mathrm{Proj}/A) \hookrightarrow \mathbf{K}(\mathrm{Flat}/A)$ is obtained using Brown representability.

Proposition 61. The category $\mathbf{K}(\mathrm{Proj}/A)$ is compactly generated, and we obtain the diagram

$$\begin{array}{ccccc} \mathbf{K}(\mathrm{Proj}/A) & \xrightarrow{\mathrm{Hom}_A(-,A)} & \mathbf{K}(\mathrm{Mod}/A) & \twoheadrightarrow & \mathbf{D}(\mathrm{Mod}/A) \\ \uparrow & & & & \uparrow \\ \mathbf{K}(\mathrm{Proj}/A)^{\mathrm{c}} & & \xrightarrow{\cong} & & \mathbf{D}^{\mathrm{fin}}(A) \end{array} \tag{41}$$

where the compact objects are not quite the projective resolutions of finitely generated modules, but rather their duals.

This will yield the "infinite completion" of Grothendieck duality (in the affine case):

Theorem 62 (Iyengar–Krause)**.** The functor $- \otimes_A D_A \colon \mathbf{K}(\mathrm{Proj}/A) \to \mathbf{K}(\mathrm{Inj}/A)$ is an equivalence.

The passage to the compacts gives the relationship to the usual duality

$$
\begin{array}{ccc}
\mathbf{K}(\mathrm{Proj}/A)^{\mathrm{c}} & \xrightarrow{-\otimes_A D_A} & \mathbf{K}(\mathrm{Inj}/A)^{\mathrm{c}} \\
{\scriptstyle \mathrm{Hom}_A(-,A)}\downarrow & & \downarrow{\scriptstyle \cong} \\
\mathbf{D}^{\mathrm{fin}}(A) & \xrightarrow{\mathbf{R}\mathrm{Hom}_A(-,D_A)} & \mathbf{D}^{\mathrm{fin}}(A)
\end{array} \tag{42}
$$

with the bottom line being Grothendieck duality in the classical sense.

Remark 63. The results of the PhD thesis by Murfet generalise this to noetherian schemes.

5 Day 5: Stratifying small triangulated categories

Whereas on day 3 we discussed the stratification of big triangulated categories we now focus on the story in the small setting, i.e. the isomorphism classes of the objects in the category actually form a set.

5.1 Example: stratifying the bounded derived category of the Kronecker algebra

Theorem 64 (Beilinson)**.** The object $T := \mathcal{O}_{\mathbb{P}^n_k} \oplus \mathcal{O}_{\mathbb{P}^n_k}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n_k}(n)$ in $\mathrm{coh}/\mathbb{P}^n_k$ is tilting, i.e. it induces an equivalence

$$\mathbf{R}\mathrm{Hom}(T,-)\colon \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/\mathbb{P}^n_k) \to \mathbf{D}^{\mathrm{b}}(\mathrm{mod}/\Lambda_n) \tag{43}$$

where Λ_n is the *Beilinson algebra* $\mathrm{End}(T)$, which is a finite-dimensional algebra.

The Beilinson algebra is described by the Beilinson quiver with relations, which is given by

$$
\overset{0}{\circ} \underset{x_n}{\overset{x_0}{\rightrightarrows}} \overset{1}{\circ} \underset{x_n}{\overset{x_0}{\rightrightarrows}} \overset{2}{\circ} \quad \cdots \quad \overset{n-1}{\circ} \underset{x_n}{\overset{x_0}{\rightrightarrows}} \overset{n}{\circ} \tag{44}
$$

and relations $x_i x_j = x_j x_i$, for all $i, j = 0, \ldots, n$.

In the special case that $n = 1$ we get the Kronecker algebra

$$\begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix} \tag{45}$$

which is moreover hereditary.

Problem We wish to describe the lattice of thick subcategories of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1_k)$.

This problem boils down to describing the indecomposables. The category is hereditary and the indecomposables are the so called "stalk complexes", i.e. up to shift we have that they are

$$\left\{\mathcal{O}_{\mathbb{P}^1_k}(i) \mid i \in \mathbb{Z}\right\} \cup \left\{\mathcal{O}_{\mathbb{P}^1_k,p^r} \mid p \text{ closed in } \mathbb{P}^1_k, r \in \mathbb{N}\right\}. \tag{46}$$

We can now set up the lattice of thick subcategories:

1. Denote the set of closed points by $\mathbb{P}^1_k(k)$ and let $\mathcal{U} \subseteq \mathbb{P}^1_k(k)$. We associate the thick subcategory

   $$\mathcal{U} \mapsto \mathcal{C}_{\mathcal{U}} := \mathrm{Thick}\left(\left\{\mathcal{O}_{\mathbb{P}^1_k,p} \mid p \in \mathcal{U}\right\}\right) \tag{47}$$

   of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1_k)$ to it.
2. Let $i \in \mathbb{Z}$. We associate the thick subcategory

   $$i \mapsto \mathcal{C}_i := \mathrm{Thick}\left(\left\{\mathcal{O}_{\mathbb{P}^1_k}(i)\right\}\right) \tag{48}$$

   of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1_k)$ to it.

These thick subcategories are ordered by inclusion, which gives us the desired lattice structure. For the thick subcategories of the first kind we moreover have that $\mathcal{U} \subseteq \mathcal{V}$ if and only if $\mathcal{C}_{\mathcal{U}} \subseteq \mathcal{C}_{\mathcal{V}}$. The $\mathcal{C}_i$ on the other hand are incomparable to eachother.

This gives us the following picture:

$$\begin{array}{ccccccccc} & \mathbf{D}^{\mathrm{b}}(\mathbb{P}^1_k) & & & \text{discrete} & & \text{continuous} \\ \diagup & & \diagdown & & & & \\ \mathcal{C}_i & & \mathcal{C}_{\mathcal{U}} & \cong & \mathbb{Z} & \sqcup & 2^{\mathbb{P}^1_k(k)} \\ \diagdown & & \diagup & & & & \\ & \{0\} & & & & & \end{array} \tag{49}$$

i.e. we take the coproduct of lattices. The exceptional objects $\mathcal{O}_{\mathbb{P}^1_k}(i)$ give rise to "discrete" information in the lattice, we have one thick subcategory for each $i \in \mathbb{Z}$ and these are all incomparable (i.e. the usual order of $\mathbb{Z}$ is not important here), whereas the subsets of closed points correspond to "continuous" information.

5.2 Stratification of bounded derived categories of hereditary algebras

This behaviour can be generalised to all path algebras, and in the Dynkin case (i.e. the underlying graph of Q is of Dynkin type $\Delta = \mathrm{A}_n, \mathrm{D}_n$ or $\mathrm{E}_{6,7,8}$) we get the following theorem (recall that $\mathrm{Z}^\bullet(\mathbf{D}^{\mathrm{b}}(kQ)) \cong k$ so we cannot use stratification by a ring action to describe the localising subcategories).

Theorem 65. There are bijections

$$\begin{array}{ccc} \{\text{localising subcategories of } \mathbf{D}(\mathrm{Mod}/A)\} & & \mathcal{C} \\ \updownarrow \scriptstyle 1:1 & & \downarrow \\ \{\text{thick subcategories of } \mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A)\} & & \mathcal{C}\cap\mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A) \\ \updownarrow \scriptstyle 1:1 & & \\ \{\text{non-crossing partitions of type } \Delta\} & & \end{array} \tag{50}$$

where non-crossing partitions are a subset of the Weyl group $\mathrm{W}(\Delta)$.

In order to describe the second bijection in this diagram we use the fact that thick subcategories of $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A)$ are all generated by exceptional objects, i.e. are described by $\mathrm{Thick}(\{E_1,\ldots,E_r\})$. Associated to E_i is a reflection s_{E_i} in the Weyl group, and the second bijection sends $\mathcal{D}$ to the composition of reflections $\mathrm{s}_{E_1}\cdots\mathrm{s}_{E_r}$. This is a result of Ingalls–Thomas.

Remark 66. The theorem generalises to arbitrary quivers, if one considers the set of all thick subcategories generated by exceptional objects instead of all thick subcategories.

In general we get the following correspondence for A a hereditary finite-dimensional k-algebra.

$\mathcal{C}\subseteq\mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A)$	$\mathcal{C}\cap\mathrm{mod}/A$
thick	thick
admissible	having a projective generator

where $\mathcal{C}\cap\mathrm{mod}/A$ denotes those objects of $\mathcal{C}$ concentrated in degree 0, and the left adjoint to the inclusion $\mathcal{C}\cap\mathrm{mod}/A\hookrightarrow\mathrm{mod}/A$ furnishes a projective generator, for which we have an exceptional collection of the module category. Hence admissible subcategories correspond to subcategories generated by exceptional sequences.

5.3 Stratification of small triangulated categories

We now focus on the local-to-global principle for small triangulated categories $\mathcal{T}$. This is based on an arXiv preprint by Benson–Iyengar–Krause.

Assume that a graded commutative ring R acts on $\mathcal{T}$.

Proposition 67. For all $\mathfrak{p}\in\operatorname{Spec}R$ there exists an exact quotient functor

(51) $\mathcal{T}\to\mathcal{T}_{\mathfrak{p}} : X\mapsto X_{\mathfrak{p}}$

such that

(52) $\mathrm{Hom}^{\bullet}_{\mathcal{T}}(X,Y)_{\mathfrak{p}}\cong\mathrm{Hom}_{\mathcal{T}_{\mathfrak{p}}}(X_{\mathfrak{p}},Y_{\mathfrak{p}})$.

This yields the notion of $\mathfrak{p}$-*local* objects.

Fix $\mathfrak{a} = (r_1, \dots, r_n)$ a homogeneous ideal of R. For any homogeneous $r \in R$, with $d = |r|$ and X an object of $\mathcal{T}$ we define

(53) $X /\!\!/ r := \operatorname{cone}(X \xrightarrow{r} \Sigma^d X)$

which is a *Koszul object*. For $\mathfrak{a} = (r_1, \dots, r_n)$ we then define

(54) $X /\!\!/ \mathfrak{a} := X_n$

where $X_0 := X$ and $X_i := X_{i-1} /\!\!/ r_i$, for $i = 1, \dots, n$.

Of course, this definition depends on the choice of generators, and moreover cones are not functorial in general. But one can obtain the following independence result.

Lemma 68. $\operatorname{Thick}(X /\!\!/ \mathfrak{a})$ is independent of the choice of generators, as

(55) $\operatorname{Thick}(X /\!\!/ \mathfrak{a}) = \left\{ Y \in \operatorname{Thick}(X) \mid \operatorname{End}^\bullet_{\mathcal{T}}(Y)_{\mathfrak{p}} = 0 \forall \mathfrak{p} \not\supseteq \mathfrak{a} \right\}.$

This yields the notion of $\mathfrak{a}$-*torsion* objects.

Definition 69. For a homogeneous prime ideal $\mathfrak{p}$ we set

(56) $X(\mathfrak{p}) := (X /\!\!/ \mathfrak{p})_{\mathfrak{p}} \cong X_{\mathfrak{p}} /\!\!/ \mathfrak{p}$

(i.e. the quotient functor and the formation of Koszul objects commute).

This yields the following local-to-global principle for thick subcategories, which is a criterion for an object to belong to a thick subcategory.

Theorem 70. For a thick subcategory $\mathcal{S}$ in $\mathcal{T}$ and $X \in \mathcal{T}$ the following are equivalent:

1. $X \in \mathcal{S}$;
2. $X_{\mathfrak{p}} \in \mathcal{S}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$;
3. $X(\mathfrak{p}) \in \mathcal{S}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

We can now define support in this situation.

Definition 71. Let X be an object of a triangulated category $\mathcal{T}$. Then the *support* of X with respect to the action of R is

(57) $\operatorname{supp}_R(X) := \{\mathfrak{p} \in \operatorname{Spec} R \mid X(\mathfrak{p}) \neq 0\}.$

Remark 72. We can also compute the support in terms of cohomology and get the following inclusion

(58) $\operatorname{supp}_R X \subseteq \left\{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{End}^\bullet_{\mathcal{T}}(X)_{\mathfrak{p}} \neq 0 \right\} = \operatorname{Supp}_R \operatorname{End}^\bullet_{\mathcal{T}}(X).$

The equality holds if $\operatorname{End}^\bullet_{\mathcal{T}}(X)$ is finitely generated over R.

For a homogeneous prime ideal $\mathfrak{p}$ of R we set

(59) $\Gamma_{\mathfrak{p}}(\mathcal{T}) := \left\{ X \in \mathcal{T}_{\mathfrak{p}} \mid \operatorname{End}^\bullet_{\mathcal{T}}(X)_{\mathfrak{q}} = 0 \forall \mathfrak{q} \not\supseteq \mathfrak{p} \right\},$

i.e. we take the $\mathfrak{p}$-local and $\mathfrak{p}$-torsion objects, which forms a thick subcategory of $\mathcal{T}_{\mathfrak{p}}$.

Definition 73. The triangulated category $\mathcal{T}$ is *stratified by the action of R* if each of the $\Gamma_{\mathfrak{p}}(\mathcal{T})$ has no proper thick subcategories.

On day 3 we had a similar result for the unbounded case. The result for small categories generalises a result from Hopkins:

Example 74. For a commutative noetherian ring A we have that $\mathbf{D}^{\mathrm{perf}}(A)$ is stratified by $R = A$.

5.4 Consequences of stratification

Theorem 75. Suppose that $\mathcal{T}$ is stratified by R. Then for all objects X and Y of $\mathcal{T}$ we have that

1. $X \in \operatorname{Thick}(Y)$ if and only if $\operatorname{supp}_R X \subseteq \operatorname{supp}_R Y$;
2. $\operatorname{Hom}^\bullet_{\mathcal{T}}(X, Y) = 0$ if and only if $\operatorname{supp}_R X \cap \operatorname{supp}_R Y = \emptyset$.

Remark that (2) in the theorem relates an asymmetric condition to a symmetric one, which is an important obstruction to having a stratification.

We can also prove a converse to this theorem:

Proposition 76. Assume that $\operatorname{End}^\bullet_{\mathcal{T}}(X)$ is finitely generated over R for all objects X of $\mathcal{T}$. If $\mathcal{T}$ is not stratified then there exists a pair of objects X and Y in $\mathcal{T}$ such that $\operatorname{supp}_R X = \operatorname{supp}_R Y$ but $\operatorname{Thick}(X) \neq \operatorname{Thick}(Y)$.

As a final remark: the results of day 3 depend on localisation techniques which are based on the so-called infinite methods, such as homotopy colimits and Brown representability, whereas today's results use categories of cohomological functors to make things work.

6 References to the literature

Day 1: [3, 11, 13]

Day 2: [4, 7, 12]

Day 3: [1, 2]

Day 4: [6, 8, 10, 14]

Day 5: [5, 9]

References

[1] D. J. Benson, S. B. Iyengar, and H. Krause, Local cohomology and support for triangulated categories, Ann. Scient. Éc. Norm. Sup. (4) **41** (2008), 1–47.

[2] D. Benson, S. B. Iyengar and H. Krause, Stratifying triangulated categories, J. Topol. **4** (2011), no. 3, 641–666.

[3] D. J. Benson, S. B. Iyengar and H. Krause, Colocalizing subcategories and cosupport, J. Reine Angew. Math. **673** (2012), 161–207.

[4] D. J. Benson, S. Iyengar and H. Krause, *Representations of finite groups: local cohomology and support*, Oberwolfach Seminars, 43, Birkhäuser/Springer Basel AG, Basel, 2012.

[5] D. J. Benson, S. B. Iyengar and H. Krause, A local-global principle for small triangulated categories, arXiv:1305.1668 (to appear in Math. Proc. Camb. Phil. Soc.).

[6] S. Iyengar and H. Krause, Acyclicity versus total acyclicity for complexes over Noetherian rings, Doc. Math. **11** (2006), 207–240 (electronic).

[7] S. B. Iyengar and H. Krause, The Bousfield lattice of a triangulated category and stratification, Math. Z. **273** (2013), no. 3-4, 1215–1241.

[8] H. Krause, The stable derived category of a noetherian scheme, Compositio Math. **141** (2005), 1128–1162.

[9] H. Krause, Report on locally finite triangulated categories, J. K-Theory **9** (2012), no. 3, 421–458.

[10] D. Murfet, The Mock Homotopy Category of Projectives and Grothendieck Duality, PhD thesis, Australian National University, 2007.

[11] A. Neeman, The chromatic tower for $\mathbf{D}(R)$, Topology **31** (1992), 519–532.

[12] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies 148, Princeton University Press, 2001.

[13] A. Neeman, Colocalizing subcategories of $\mathbf{D}(R)$, J. Reine Angew. Math. **653** (2011), 221–243.

[14] G. Stevenson, Subcategories of singularity categories via tensor actions, arXiv:1105.4698 (to appear in Compositio Math.).