# Dispersive estimates for magnetic Schrödinger operators

Hynek Kovařík (Università di Brescia)

Magnetic fields and semi-classical analysis

Rennes, May 21, 2015

# General setting

Let $B : \mathbb{R}^2 \to \mathbb{R}$ be a magnetic field and consider the Schrödinger operator $H(B)$ in $L^2(\mathbb{R}^2)$ formally given by

$$H(B) = (i\nabla + A)^2$$

where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $|A| \in L^2_{loc}(\mathbb{R}^2)$ and curl $A = B$ holds in the distributional sense.

- We will work under the condition $|A| \in L^\infty(\mathbb{R}^2)$; hence we define $H(B)$ as the unique self-adjoint operator associated with the closed quadratic form

$$Q[u] = \int_{\mathbb{R}^2} |(i\nabla + A)\, u|^2 \, dx, \qquad d(Q) = W^{1,2}(\mathbb{R}^2).$$

# General setting

Obviously, $H(B) \geq 0$. We assume that $B$ is such that

$$\sigma(H(B)) = [0, \infty).$$

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a bounded electric potential with a suitable decay at infinity such that $\sigma_{es}(H(B) + V) = [0, \infty)$.

- The problem: we want to study the influence of the magnetic on the asymptotic behavior of the solutions to the Schrödinger equation

$$i\,\partial_t u = (H(B) + V)\,u$$

# General setting

Hence the object our interest is the unitary group $e^{-it(H(B)+V)}$

In particular, we want to compare the time decay of

$$e^{-it(H(B)+V)}\, P_c^B \qquad \text{as} \qquad t \to +\infty$$

where $P_c^B$ is the projection onto the continuous subspace of $L^2(\mathbb{R}^2)$ with respect to $H(B) + V$, with the decay of its non-magnetic counterpart:

$$e^{-it(-\Delta+V)}\, P_c \qquad \text{as} \qquad t \to +\infty$$

Here $P_c$ is the projection onto the continuous subspace of $L^2(\mathbb{R}^2)$ with respect to $-\Delta + V$

# Time decay: non-magnetic Schrödinger operators

- $L^1 \to L^\infty$ estimates: one considers the propagator $e^{-it(-\Delta+V)}\, P_c$ as operator from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ and studies the time decay of the corresponding norm

$$\| e^{-it(-\Delta+V)}\, P_c \|_{L^1 \to L^\infty}$$

- If $V = 0$, then

$$e^{it\Delta}(x,y) = (4\, i\, \pi\, t)^{-n/2}\, e^{\frac{i\, |x-y|^2}{4t}}, \qquad x, y \in \mathbb{R}^n$$

Hence

$$\| e^{it\Delta} \|_{L^1 \to L^\infty} \ \leq \ (4\, \pi\, t)^{-\frac{n}{2}} \qquad t > 0.$$

# Time decay: non-magnetic Schrödinger operators

An alternative, thought less precise, way to measure the time decay is to consider $e^{-it(-\Delta+V)}$ as an operator between weighted $L^2-$spaces;

$$e^{-it(-\Delta+V)}\, P_c \ : \ L^2(\mathbb{R}^n, \rho^2\, dx) \to L^2(\mathbb{R}^n, \rho^{-2}\, dx),$$

or equivalently

$$\rho^{-1}\, e^{-it(-\Delta+V)}\, P_c\, \rho^{-1} \ : \ L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

where $\rho > 0$ is a suitable weight function.

For $V = 0$ the Cauchy-Schwarz inequality gives

$$\|\, \rho^{-1}\, e^{it\Delta}\, \rho^{-1}\, u\, \|_{L^2(\mathbb{R}^n)} \ \lesssim \ t^{-\frac{n}{2}}\, \|\rho^{-1}\,\|^2_{L^2(\mathbb{R}^n)}\, \|u\|_{L^2(\mathbb{R}^n)}$$

provided

$$\rho(x) = (1+|x|)^{\frac{n}{2}+\varepsilon}, \quad \varepsilon > 0.$$

# Time decay: non-magnetic Schrödinger operators

- If $V \neq 0$, then the decay rate depends on the validity of the estimate

$$\limsup_{z\to 0} \| \, \rho^{-1} \, (-\Delta + V - z)^{-1} \, \rho^{-1} \|_{2\to 2} \; < \; \infty \tag{1}$$

If (1) holds true for some $\rho$, then we say that zero is a regular point of $-\Delta + V$; (generic situation).

- Zero is not a regular point of $-\Delta$ in $L^2(\mathbb{R}^n)$ for $n = 1, 2$.

- Zero is a regular point of $-\Delta$ in $L^2(\mathbb{R}^n)$ for $n \geq 3$:

$$\limsup_{z\to 0} \| \, \rho^{-1} \, (-\Delta - z)^{-1} \, \rho^{-1} \|_{2\to 2} \; < \; \infty$$

if $\rho(x) = (1 + |x|)^\beta$, with $\beta \geq 1$.

# Time decay: non-magnetic Schrödinger operators

- Dimension $n = 3$. If zero is a regular point of $-\Delta + V$, then as $t \to \infty$

$$\| \rho^{-1} e^{-it(-\Delta+V)} P_c \rho^{-1} \|_{2\to 2} = \mathcal{O}\big(t^{-\frac{3}{2}}\big) \tag{2}$$

**[Rauch 1978]**: $\rho(x) = e^{\varepsilon|x|}$ and $V(x) \lesssim e^{-\varepsilon|x|}$, $\varepsilon > 0$.

**[Jensen-Kato 1979]**: $\rho(x) = (1 + |x|)^{\beta}$, $\beta > 5/2$, and $V(x) \lesssim (1 + |x|)^{-3}$.

**[Journeé-Soffer-Sogge 1991, Goldberg-Schlag 2004, Goldberg 2006]** ....

If zero is not a regular point of $-\Delta + V$, then (2) fails and one observes a slower decay: **[Rauch 1978, Jensen-Kato 1979, Murata 1982]**

# Time decay: non-magnetic Schrödinger operators

- Dimension $n = 2$. **[Schlag 2005]** : if zero is a regular point of $-\Delta + V$, then

$$\| \, \rho^{-1} \, e^{-it(-\Delta+V)} \, P_c \, \rho^{-1} \, \|_{2\to 2} \; = \; \mathcal{O}(t^{-1}) \qquad t \to \infty. \tag{3}$$

holds for $\rho(x) = (1+|x|)^\beta, \;\beta > 1$ and $V(x) \lesssim (1+|x|)^{-3}$. This is again the decay rate of the free evolution. However, (3) **can be improved**, still under the condition that zero is a regular point, provided $\rho$ grows fast enough:

$$\| \, \rho^{-1} \, e^{-it(-\Delta+V)} \, P_c \, \rho^{-1} \, \|_{2\to 2} \; = \; \mathcal{O}(t^{-1} \, (\log t)^{-2}) \qquad t \to \infty \tag{4}$$

where $\rho(x) = (1+|x|)^\beta, \;\beta > 3$, and $V(x) \lesssim (1+|x|^2)^{-3}$, **[Murata 82]**, see also **[Goldberg-Green 2013]**.

- Hence adding a potential $V$ might improve the decay rate, contrary to the case $n \geq 3$.

# Time decay: magnetic Schrödinger operators

■ Dimension $n = 3$. **[Murata: 1982]** showed, under suitable regularity and decay assumptions on $B$ and $V$, that if zero is a regular point of $H(B) + V$, and $\rho(x) = (1 + |x|)^\beta$ with $\beta$ large enough, then

$$\| \, \rho^{-1} \, e^{-it(H(B)+V)} \, P_c \, \rho^{-1} \, \|_{2\to 2} \; = \mathcal{O}(t^{-3/2}) \qquad t \to \infty \tag{5}$$

Moreover it follows from **[Murata: 1982]** hat the decay rate in (5) is **sharp**. Hence a magnetic field, decaying at infinity, **does not improve the decay rate** of $e^{-it(H(B)+V)}$ in dimension three.

■ Dimension $n = 2$. Our motivation is to show that a compactly supported magnetic field in dimension two **does improve** the decay of $e^{-it(H(B)+V)}$ as $t \to \infty$ and that the decay rate is given by its **total flux**.

# Main results: weighted $L^2-$estimates

- Assumption 1: Let $B \in C^\infty(\mathbb{R}^2; \mathbb{R})$ be such that for some $\sigma > 4$ we have

$$\sup_{\theta \in (0,2\pi)} \big( |B(r,\theta)| + |\partial_\theta B(r,\theta)| \big) \lesssim (1+r)^{-\sigma}.$$

Under this assumption we can define the following quantities:

$$\alpha = \frac{1}{2\pi} \int_{\mathbb{R}^2} B(x)\, dx < \infty, \qquad \mu(\alpha) := \min_{k \in \mathbb{Z}} |k - \alpha| \in [0, 1/2]\,.$$

- Assumption 2: Let $V : \mathbb{R}^2 \to \mathbb{R}$ be bounded and such that the operator $H(B) + V$ has no positive eigenvalues.

- $\sigma_{es}(H(B)+V) = \sigma_c(H(B)+V) = [0, \infty)$.

# Main results: weighted $L^2-$estimates

**Theorem (K.):** Let $\alpha \notin \mathbb{Z}$. Put $\rho(x) = (1+|x|)^s$ with $s > 5/2$ and suppose that $|V(x)| \lesssim (1+|x|)^{-3}$. If zero is a regular point of $H(B)+V$, then there exists an operator

$$K(B,V) \in \mathscr{B}(L^2(\mathbb{R}^2))$$

such that

$$\boxed{\rho^{-1}\, e^{-it(H(B)+V)}\, P_c^B\, \rho^{-1} \;=\; t^{-1-\mu(\alpha)}\, K(B,V) + o(t^{-1-\mu(\alpha)})}$$

in $\mathscr{B}(L^2(\mathbb{R}^2))$ as $t \to \infty$.

# Main results: weighted $L^2-$estimates

- The maximal decay rate $t^{-3/2}$, for $\mu(\alpha) = 1/2$, is the same as in dimension three.

- The operator $K(B, V)$ can be expressed explicitly in terms of $B$ and $V$. Its $L^2-$norm is gauge-invariant.

- If $\rho(x) = (1 + |x|)^\beta$ then we must have $\beta \geq 1$.

- If $V = 0$, then zero is a regular point of $H(B)$:

$$\frac{1}{1 + |x|^2} \lesssim H(B)$$

in the sense of quadratic forms on $W^{1,2}(\mathbb{R}^2)$; **[Laptev-Weidl 1999]**.

# Main results: weighted $L^2-$estimates

**Theorem (K.):** Let $\alpha \in \mathbb{Z}$. Put $\rho(x) = (1+|x|)^s$ with $s > 5/2$ and suppose that $|V(x)| \lesssim (1+|x|)^{-3}$. If zero is a regular point of $H(B)+V$, then there exists an operator

$$\widetilde{K}(B,V) \in \mathscr{B}(L^2(\mathbb{R}^2))$$

such that

$$\boxed{\rho^{-1}\, e^{-it(H(B)+V)}\, P_c^B\, \rho^{-1} \;=\; t^{-1}(\log t)^{-2}\, \widetilde{K}(B,V) + o(t^{-1}(\log t)^{-2})}$$

in $\mathscr{B}(L^2(\mathbb{R}^2))$ as $t \to \infty$.

# Main ingredients of the proof

Assume that $\alpha \notin \mathbb{Z}$ and that $V = 0$.

By the spectral theorem and Stone formula we have

$$\rho^{-1}\, e^{-itH(B)}\, \rho^{-1} \; = \int_0^\infty e^{-it\lambda}\, E(\alpha,\lambda)\, d\lambda, \tag{6}$$

where $E(\alpha, \lambda)$ is the (weighted) spectral density associated to $H(B)$:

$$E(\alpha,\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon\to 0+} \rho^{-1} \left[(H(B)-\lambda-i\varepsilon)^{-1} - (H(B)-\lambda+i\varepsilon)^{-1}\right] \rho^{-1}$$

We will use the notation

$$R_+(\alpha,\lambda) = \lim_{\varepsilon\to 0+} (H(B)-\lambda-i\varepsilon)^{-1}$$

# Main ingredients of the proof

Let $\phi \in C^\infty(0,\infty)$, $0 \leq \phi \leq 1$, be such that $\phi(x) = 0$ for $x$ large enough and $\phi(x) = 1$ in a neighborhood of $0$.

$$\int_0^\infty e^{-it\lambda} E(\alpha,\lambda)\, d\lambda = \int_0^\infty e^{-it\lambda} (1-\phi)\, E(\alpha,\lambda)\, d\lambda + \int_0^\infty e^{-it\lambda} \phi\, E(\alpha,\lambda)\, d\lambda$$

Our aim is to show that

$$\int_0^\infty e^{-it\lambda} (1-\phi(\lambda))\, E(\alpha,\lambda)\, d\lambda \; = \; o(t^{-2})$$

and

$$\int_0^\infty e^{-it\lambda} \phi(\lambda)\, E(\alpha,\lambda)\, d\lambda \; = \; t^{-1-\mu(\alpha)}\, K(B,V) + o(t^{-1-\mu(\alpha)})$$

in $\mathscr{B}(L^2(\mathbb{R}^2))$ as $t \to \infty$.

# Main ingredients of the proof

We need to prove that

$$E(\alpha, \lambda) = E_1 \, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \qquad \lambda \to 0$$

for some $E_1 \in \mathscr{B}(L^2(\mathbb{R}^2))$. We have to show that

$$\rho^{-1} \, R_+(\alpha, \lambda) \, \rho^{-1} \; = \; F_0 + F_1 \, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \qquad \lambda \to 0.$$

- Recall that in the absence of magnetic field we have

$$\rho^{-1} \, R_+(\lambda) \, \rho^{-1} \; = \widetilde{F}_0 \, \log \lambda + \mathcal{O}(1) \qquad \lambda \to 0.$$

# Resolvent expansion at threshold

Consider a radial magnetic field $B_0$ generated by the vector potential

$$A_0(x) = \alpha\,(-x_2, x_1) \begin{cases} |x|^{-1} & |x| \le 1 \\ |x|^{-2} & |x| > 1 \end{cases} \qquad \nabla \cdot A_0 = 0.$$

$$B_0(x) = \operatorname{curl} A_0(x) = \begin{cases} \alpha\,|x|^{-1} & |x| \le 1 \\ \\ 0 & |x| > 1 \end{cases}, \qquad \frac{1}{2\pi} \int_{\mathbb{R}^2} B_0(x)\,dx = \alpha.$$

Using the partial wave decomposition, after some calculations we find that

$$\rho^{-1}\ R_+^0(\alpha, \lambda)\ \rho^{-1} \ = \ G_0 + G_1\,\lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \quad \lambda \to 0$$

for some $G_0, G_1$ in $\mathscr{B}(L^2(\mathbb{R}^2))$, where $R_+^0(\alpha, \lambda)$ is the resolvent of $H(B_0)$.

# Resolvent expansion at threshold

**Lemma**: Let $\alpha > 0$ be the flux of $B$ through $\mathbb{R}^2$. Then there exists a bounded vector field $A = (a_1, a_2)$ s. t. curl $A = \partial_1 a_2 - \partial_2 a_1 = B$ in the distributional sense, and

$$|\nabla \cdot A(x)| \;=\; o\left(|x|^{-3}\right) \;, \qquad |\, A(x) - A_0(x)| \;=\; o\left(|x|^{-3}\right)$$

- The above Lemma implies that

$$T(B) := H(B) - H(B_0) = 2\,i\,\underbrace{(A - A_0)}_{o\left(|x|^{-3}\right)} \cdot \nabla \;+\; \underbrace{i\,\nabla \cdot A}_{o\left(|x|^{-3}\right)} \;+\; \underbrace{|A|^2 - |A_0|^2}_{o\left(|x|^{-3}\right)}$$

since $\nabla \cdot A_0 = 0$. This allows us to show that the operator

$$G_0\, \rho\, T(B)\, \rho = \rho^{-1}\, H(B_0)^{-1}\, T(B)\, \rho$$

is compact in $\mathscr{B}(L^2(\mathbb{R}^2))$.

# Resolvent expansion at threshold

With this we prove that $1 + G_0\, \rho\, T(B)\, \rho$ is invertible in $L^2(\mathbb{R}^2)$. Then

$$1 + \rho^{-1} R_+^0(\alpha, \lambda)\, T(B)\, \rho = 1 + G_0\, \rho\, T(B)\, \rho + G_1\, \rho\, T(B)\, \rho\, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)})$$

is invertible for $\lambda$ small enough. From the resolvent equation we thus obtain

$$\rho^{-1}\, R_+(\alpha, \lambda)\, \rho^{-1} = \big(1 + \rho^{-1} R_+^0(\alpha, \lambda)\, T(B)\, \rho\big)^{-1} \rho^{-1}\, R_+^0(\alpha, \lambda)\, \rho^{-1}$$

Since

$$\big(1 + \rho^{-1} R_+^0(\alpha, \lambda)\, T(B)\, \rho\big)^{-1} = (1 + G_0\, \rho\, T(B)\, \rho)^{-1} + S(B)\, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}),$$

we arrive at

$$\rho^{-1}\, R_+(\alpha, \lambda)\, \rho^{-1} = F_0 + F_1\, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \qquad \lambda \to 0.$$

# Remark

- In order that the coefficients of $H(B_1, V) - H(B_2, V)$ decay faster than $o(|x|^{-1})$ at infinity, the fluxes of $B_1$ and $B_2$ must be equal.

Indeed, if $\operatorname{curl} A_1 = B_1$ and $\operatorname{curl} A_2 = B_2$, then by the Stokes Theorem we have

$$|A_1(x) - A_2(x)| = o(|x|^{-1}) \quad |x| \to \infty \quad \Rightarrow \quad \int_{\mathbb{R}^2} B_1(x)\, dx = \int_{\mathbb{R}^2} B_2(x)\, dx.$$

# $L^1 \to L^\infty$ estimates: scaling critical Schrödinger operators

We consider Schrödinger operators in $L^2(\mathbb{R}^n)$, $n \geq 2$ of the form

$$H(A,a) = \left(-i\nabla + |x|^{-1}\, A\left(\frac{x}{|x|}\right)\right)^2 + |x|^{-2}\, a\Big(\frac{x}{|x|}\Big)\,,$$

where $A \in W^{1,\infty}(\mathbb{S}^{n-1}, \mathbb{R}^n)$, $a \in W^{1,\infty}(\mathbb{S}^{n-1}, \mathbb{R})$ and $\mathbb{S}^{n-1}$ denotes the $n-$dimensional unit sphere.

Under the scaling $x \mapsto \lambda\, x$ we have

$$H(A,a) \mapsto \lambda^{-2}\, H(A,a).$$

We are interested in the unitary group $e^{-itH(A,a)}$ generated by $H(A,a)$.

# $L^1 \to L^\infty$ estimates: scaling critical Schrödinger operators

The behaviour of $e^{-itH(A,a)}$ is closely related to the spectral properties of the operator

$$L(A,a) = (-i\nabla_{\mathbb{S}^{n-1}} + A)^2 + a \qquad \text{in} \quad L^2(\mathbb{S}^{n-1}),$$

where $\nabla_{\mathbb{S}^{n-1}}$ denotes the spherical gradient on $\mathbb{S}^{n-1}$. If $A \equiv a \equiv 0$, then $L(A,a)$ coincides with the Laplace-Beltrami operator on $L^2(\mathbb{S}^{n-1})$. The spectrum of $L(A,a)$ is purely discrete.

We denote by $\{\lambda_k(A,a)\}$ and $\{\psi_k\}$ the sequences of its eigenvalues and normalized eigenfunctions:

$$L(A,a)\,\psi_k = \lambda_k(A,a)\,\psi_k, \qquad \|\psi_k\|_{L^2(\mathbb{S}^{n-1})} = 1.$$

# $L^1 \to L^\infty$ estimates: scaling critical Schrödinger operators

**Theorem (Fanelli, Grillo, K.):** Let $n \geq 2$ and assume that $\lambda_1(A,a) \geq 0$. Denote by

$$g(n) = \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(A,a)} \, - \frac{n-2}{2} \, \geq 0 \, .$$

If, for all $t > 0$ and some $C_0$, the following estimate holds

$$\| \, e^{-itH(A,a)} \, \|_{L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \; \leq \; C_0 \; t^{-\frac{n}{2}} \; ,$$

then there exists a constant $C$ such that

$$\boxed{\Big\| \, |x|^{-g(n)} \; e^{-itH(A,a)} \, |x|^{-g(n)} \, \Big\|_{L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \; \leq \; C \; t^{-\frac{n}{2} - g(n)}}$$

holds for all $t > 0$.

# $L^1 \to L^\infty$ estimates: scaling critical Schrödinger operators

- One of the main ingredients of the proof is the representation formula for the integral kernel of $e^{-itH(A,a)}$ which was found by **[Fanelli-Felli-Fontelos-Primo, 14]**, for any $u_0 \in C_0^\infty(\mathbb{R}^n)$ we have

$$\big(e^{-itH(A,a)}\, u_0\big)(x) = -\frac{i\, e^{\frac{i|x|^2}{4t}}}{(2t)^{n/2}} \int_{\mathbb{R}^n} K\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big)\, e^{\frac{i|y|^2}{4t}}\, u_0(y)\, dy,$$

where

$$K(x,y) = (|x|\,|y|)^{\frac{2-n}{2}} \sum_{k\in\mathbb{Z}} i^{-\beta_k}\, J_{\beta_k}(|x||y|)\ \psi_k\Big(\frac{x}{|x|}\Big)\, \overline{\psi_k\Big(\frac{y}{|y|}\Big)}\ ,$$

and

$$\alpha_k = \frac{n-1}{2} - \sqrt{\Big(\frac{n-2}{2}\Big)^2 + \lambda_k(A,a)}\,, \qquad \beta_k = \sqrt{\Big(\frac{n-2}{2}\Big)^2 + \lambda_k(A,a)}$$

# $L^1 \to L^\infty$ estimates: Aharonov-Bohm operator

If we put $n = 2$, $a = 0$ and

$$A(x) = A_{ab}(x) = \frac{\alpha}{|x|^2}\,(-x_2\,,\,x_1)\,, \qquad n = 2$$

then the operator $H(A_{ab}, 0)$ describes the energy of a particle interacting with the so-called Aharonov-Bohm magnetic field of flux $\alpha$ in $\mathbb{R}^2$.

Since

$$\|e^{-itH(A_{ab},0)}\|_{L^1(\mathbb{R}^2)\to L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{t} \qquad \forall\, t > 0, \quad n = 2.$$

holds true by **[Fanelli-Felli-Fontelos-Primo, 14]**, the above Theorem implies

# $L^1 \to L^\infty$ estimates: Aharonov-Bohm operator

**Corollary (Fanelli, Grillo, K.):** Let $n = 2$. Then

$$\big\| \, |x|^{-\mu(\alpha)} \; e^{-itH(A_{ab},0)} \, |x|^{-\mu(\alpha)} \, \big\|_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \; \leq \; C \; t^{-1-\mu(\alpha)}$$

holds for all $t > 0$.

- For $\alpha \in \mathbb{Z}$ we have $\mu(\alpha) = 0$ and the above equation turns into

$$\big\| e^{-itH(A_{ab},0)} \big\|_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \; \leq \; C \; t^{-1}$$

which is the decay rate of the free evolution; $H(A_{ab}, 0) \simeq -\Delta$ if $\alpha \in \mathbb{Z}$.

## $L^1 \to L^\infty$ estimates: Schrödinger operators with inverse square potentials

Consider the case $n = 3$, $A = 0$ and

$$a(x) = \frac{\beta}{|x|^2}, \qquad \beta > 0.$$

so that

$$H(0,a) = -\Delta + \frac{\beta}{|x|^2}, \qquad \beta > 0.$$

Then, again by [Fanelli-Felli-Fontelos-Primo,14] we have

$$\|e^{-itH(0,a)}\|_{L^1(\mathbb{R}^3)\to L^\infty(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}} \qquad \forall\, t > 0, \quad n = 3.$$

Hence the Theorem above gives

# $L^1 \to L^\infty$ estimates: Schrödinger operators with inverse square potentials

**Corollary (Fanelli, Grillo, K.):** Let $n = 3$ and let

$$H(0,a) = -\Delta + \frac{\beta}{|x|^2}\,, \qquad \beta > 0.$$

Then

$$\big\| \, |x|^{-\gamma} \, e^{-itH(0,a)} \, |x|^{-\gamma} \, \big\|_{L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)} \; \leq \; C \, t^{-\frac{3}{2}-\gamma}$$

where

$$\gamma = \sqrt{\frac{1}{4} + \beta} - \frac{1}{2}\,.$$

# References


- G. Grillo, H. Kovařík: Weighted dispersive estimates for two-dimensional Schrödinger operators with Aharonov-Bohm magnetic field. *J. Diff. Equations* **256** (2014) 3889–3911.

- H. Kovařík: Resolvent expansion and time decay of the wave functions for two-dimensional magnetic Schrödinger operators. *Comm. Mat. Phys.* **337** (2015) 681–726.

- L. Fanelli, G. Grillo, H. Kovařík: Improved time-decay for a class of scaling critical electromagnetic Schrödinger flows. arXiv: 1502.04987.