The Ginzburg-Landau model in the surface superconductivity regime

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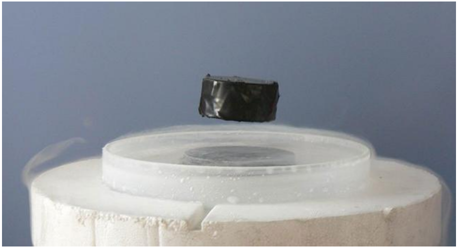
Joint work with Michele Correggi (Rome 3).

1. **Ginzburg-Landau theory of type II superconductors**
2. Surface superconductivity
3. Leading order results between H_{c2} and H_{c3}
4. Elements of proof
5. Expansion beyond the leading order

Superconductors in magnetic fields

- Superconductivity = absence of resistivity at low temperature in some materials
- Peculiar response to applied magnetic fields = small fields do not penetrate (Meissner effect)
- Ginzburg-Landau 50 : phenomenological theory, order parameter
- Bardeen-Cooper-Schrieffer 57 : microscopic theory, Cooper pairing
- Gor'kov 59: BCS $\Rightarrow$ GL, mathematically rigorous derivation Frank-Hainzl-Seiringer-Solovej 12



Superconductor levitating above a magnet

Ginzburg-Landau theory

Sample = infinite cylinder of smooth cross-section $\Omega \subset \mathbb{R}^2$, in a uniform external magnetic field perpendicular to Ω.

- Order parameter $\Psi : \mathbb{R}^2 \to \mathbb{C}$. $|\Psi|^2$ = relative density of superconducting electrons (bound in Cooper pairs)
- Induced magnetic field $h \neq$ applied magnetic field h_{ex}
- Induced magnetic vector potential $\mathbf{A}$ with curl $\mathbf{A} = h$.
- κ = penetration depth. $\kappa\sigma$ = strength of applied magnetic field
- Type II superconductor : $\kappa > 1/\sqrt{2}$, "extreme type II": $\kappa \to \infty$

Energy functional to be minimized:

$$\boxed{\mathcal{G}^{\mathrm{GL}}_{\kappa,\sigma}[\Psi, \mathbf{A}] = \int_\Omega |(\nabla + i\kappa\sigma\mathbf{A})\,\Psi|^2 - \kappa^2|\Psi|^2 + \tfrac{1}{2}\kappa^2|\Psi|^4 + (\kappa\sigma)^2\,|\mathrm{curl}\,\mathbf{A} - 1|^2}$$

Gauge invariance: energy invariant under

$$\Psi \to \Psi e^{-i\kappa\sigma\varphi}, \quad \mathbf{A} \to \mathbf{A} + \nabla\varphi$$

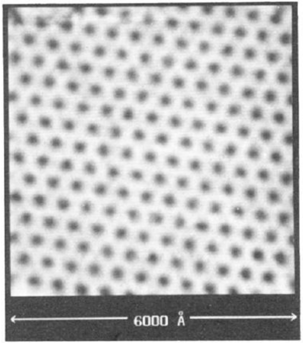
Phenomenology of type II superconductors

For minimizers $|\Psi| \leq 1$.

- $|\Psi| = 1$: purely superconducting state, all electrons in Cooper pairs.
- $|\Psi| = 0$: normal state, no Cooper pairs.
- Low magnetic field, $\kappa\sigma \leq H_{c1}$: superconducting state $|\Psi| \approx 1$ a.e.
- First critical field:
$$\kappa\sigma = H_{c1} \approx C_\Omega \log \kappa$$
isolated normal regions (vortices) start to appear.
- $H_{c1} \leq \kappa\sigma \leq H_{c2}$: vortex lattice state, Abrikosov lattice.
- Second critical field:
$$\kappa\sigma = H_{c2} \approx \kappa^2$$
superconductivity disappears uniformly in the bulk.
- $H_{c2} \leq \kappa\sigma \leq H_{c3}$: surface superconductivity state, $|\Psi| \approx 0$ in the bulk, $|\Psi| > 0$ close to the boundary.
- Normal state $|\Psi| \equiv 0$ above the third critical field:
$$\kappa\sigma > H_{c3} \approx \Theta_0^{-1}\kappa^2, \qquad \Theta_0 < 1.$$

Mixed state: Abrikosov lattice

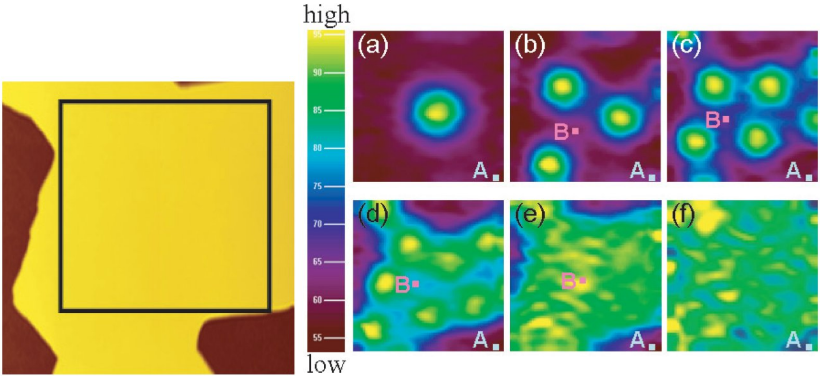
- Theoretical prediction: Abrikosov 57, first observation 67.
- External magnetic field penetrates in small normal regions.
- Mathematical literature: cf Sandier-Serfaty's 2007 book.



Vortex lattice in a type II superconductor, Hess-et al-Waszczak 89.

Mixed state: surface superconductivity

- Theoretical prediction: Saint-James and de Gennes 63, observed 64.
- Bulk is normal, magnetic field penetrates.
- A thin superconducting layer survives along the boundary.
- Mathematical literature: cf Fournais-Helffer's 2010 book.



Superconductivity in increasing magnetic fields, Ning-et al-Xue 09.

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Transition from the normal state in decreasing fields

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_\Omega \left| \left(\nabla + i\varepsilon^{-2}\mathbf{A} \right) \Psi \right|^2 + \frac{1}{2b\varepsilon^2} \left(|\Psi|^4 - 2|\Psi|^2 \right) + \frac{b}{\varepsilon^4} \left| \mathrm{curl}\, \mathbf{A} - 1 \right|^2 .$$

- New parameters: $\sigma = b\kappa$, b fixed, $\varepsilon = (\sigma\kappa)^{-1/2} \ll 1$.
- Correspondence: $H_{c2} \leftrightarrow b = 1$, $H_{c3} \leftrightarrow b = \Theta_0^{-1}$
- St-James/de Gennes 63: Start at large b, normal state $|\Psi| \equiv 0, \mathrm{curl}\, \mathbf{A} \equiv 1$. When does this become unstable ?
- At first, curl $\mathbf{A}$ stays fixed $\equiv 1$. Choice of gauge $\mathbf{A} \approx \mathbf{F}$

$$\begin{cases} \mathrm{curl}\, \mathbf{F} = 1 \text{ in } \Omega \\ \mathrm{div}\, \mathbf{F} = 0 \text{ in } \Omega \\ \nu . \mathbf{F} = 0 \text{ on } \partial\Omega \end{cases}$$

- Close to transition, for small values of Ψ, energy to leading order

$$\int_\Omega \left| \left(\nabla + i\varepsilon^{-2}\mathbf{F} \right) \Psi \right|^2 - \frac{1}{b\varepsilon^2} |\Psi|^2$$

- Can one make this < 0, smaller than energy of the normal state ?

The critical fields H_{c2} and H_{c3}

$$\mathcal{E}[\Psi] = \left\langle \Psi \middle| H_\varepsilon - \frac{1}{b\varepsilon^2} \middle| \Psi \right\rangle$$

- $H_\varepsilon = -\left(\nabla + i\varepsilon^{-2}\mathbf{F}\right)^2$, magnetic Laplacian, uniform field $= \varepsilon^{-2}$.
- When does H_ε have an eigenvalue strictly less than $1/(b\varepsilon^2)$?
- Eigenfunctions of H_ε are localized over length scales of order ε

$$\begin{cases} \text{localization in the bulk} \rightsquigarrow \text{ magnetic Laplacian in the plane} \\ \text{localization close to boundary} \rightsquigarrow \text{ magnetic Laplacian in a half-plane} \end{cases}$$

- First eigenvalues for small ε (semi-classics, e.g. Helffer-Morame)

$$\begin{cases} \text{magnetic Laplacian in the plane} \to \lambda_1 \sim \varepsilon^{-2} \\ \text{magnetic Laplacian in a half-plane} \to \lambda_1 \sim \Theta_0 \varepsilon^{-2} < \varepsilon^{-2} \end{cases}$$

- Third critical field: if $1 < b < \Theta_0^{-1}$, favorable to put mass close to the boundary, but only there.
- Second critical field: if $b < 1$, favorable to also put mass in the bulk.

More precise effective model between H_{c2} and H_{c3}

- $1 < b < \Theta_0^{-1}$, Ψ concentrated close to boundary on length scale ε.
- Magnetic field penetrates curl $\mathbf{A} \approx 1$, choose a convenient gauge.
- In scaled boundary coordinates (s,t) (units of ε^{-1}), curvature $k(s)$

$$\int_{s=0}^{|\partial\Omega|} \int_{t=0}^{c_0|\log\varepsilon|} (1-\varepsilon k(s)t) \Big\{ |\partial_t \psi|^2 + \frac{1}{(1-\varepsilon k(s)t)^2} \left|(\varepsilon\partial_s + i a_\varepsilon(s,t))\psi\right|^2 + \frac{1}{2b}\left[|\psi|^4 - 2|\psi|^2\right] \Big\}$$

- To leading order in ε, after scaling s:

$$\mathcal{E}_{\rm hp}[\psi] = \int_{s=0}^{|\partial\Omega|\varepsilon^{-1}} \int_{t=0}^{+\infty} \left\{ \left|(\nabla - it\mathbf{e}_s)\psi\right|^2 + \frac{1}{2b}|\psi|^4 - \frac{1}{b}|\psi|^2 \right\}.$$

- Natural ansatz $\psi(s,t) = f(t)e^{-i\alpha s}$ (exact in the linear case) leads to

$$\boxed{\mathcal{E}^{\rm 1D}_{0,\alpha}[f] := \int_0^{+\infty} |\partial_t f|^2 + (t+\alpha)^2 f^2 + \frac{1}{2b}\left(f^4 - 2f^2\right)}$$

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Previously known results

- $b \to \Theta_0^{-1}$ "easier case", cf Lu-Pan, Fournais-Helffer ...
- $b \to 1^+$: transition boundary to bulk behavior, Fournais-Kachmar 09
- $b \to 1^-$, cf Almog, Sandier-Serfaty, Aftalion-Serfaty, circa 07
- X.B. Pan 02, if $1 < b < \Theta_0^{-1}$, for some implicit constant $E_b < 0$

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_b}{\varepsilon} + o(\varepsilon^{-1})$$

- Minimize $\mathcal{E}_{0,\alpha}^{\mathrm{1D}}[f] \Rightarrow$ optimal energy E_0^{1D}, phase α_0, density f_0.
  Almog-Helffer 07, Fournais-Helffer-Persson 11, for $1.25 \leq b < \Theta_0^{-1}$

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + o(\varepsilon^{-1}), \qquad |\Psi^{\mathrm{GL}}|^2 \approx f_0^2(t) \text{ in } L^2(\Omega)$$

Methods (cf Fournais-Helffer's book)

- Decay estimates à la Agmon + Magnetic field estimates (elliptic PDEs methods) $\rightsquigarrow$ boundary problem
- Linear problem has unique non degenerate ground state
- Treat non linearity "perturbatively"

New energy and density estimates

The simplified 1D limit problem gives the leading order for all field strengths between H_{c2} and H_{c3}.

Theorem (Correggi-NR 13)

Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain. For any fixed $1 < b < \Theta_0^{-1}$, in the limit $\varepsilon \to 0$, it holds

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + \mathcal{O}(1),$$

and

$$\left\| |\Psi^{\mathrm{GL}}|^2 - f_0^2\,(t) \right\|_{L^2(\Omega)} \leq C\varepsilon \ll \left\| f_0^2\,(t) \right\|_{L^2(\Omega)}.$$

- Idea of proof : don't think perturbatively around the linear problem
- Use the physics of the problem : "quantum fluid mechanics"

Uniform density estimates and degree estimates

Conjecture by Pan 02: $|\Psi^{\mathrm{GL}}|^2 \to C(b) > 0$ pointwise on $\partial\Omega$.

Theorem (Correggi-NR 14)

For any $\mathbf{r} \in \Omega$ *with* $\mathrm{dist}(\mathbf{r}, \partial\Omega) \lesssim \varepsilon$ *we have*

$$\left| \left| \Psi^{\mathrm{GL}}(\mathbf{r}) \right| - f_0\left(t\right) \right| \to 0$$

- No defects (e.g. vortices) in the surface superconductivity layer.
- Phase is well-defined along $\partial\Omega$: $\Psi^{\mathrm{GL}} = \sqrt{\rho} e^{i\varphi}$.
- Phase circulation along $\partial\Omega \leftrightarrow$ number of vortices in the bulk.

Theorem (Correggi-NR 14)

Any GL minimizer Ψ^{GL} *satisfies in the limit* $\varepsilon \to 0$

$$\frac{1}{2\pi} \int_{\partial \mathcal{B}_R} \partial_\tau \varphi = \deg\left(\Psi^{\mathrm{GL}}, \partial\Omega\right) = \frac{|\Omega|}{\varepsilon^2} + \frac{|\alpha_0|}{\varepsilon}(1 + o(1)).$$

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Preliminary reductions

- Agmon estimates $\rightarrow$ exponential decay of order parameter away from the boundary (distances $\gg \varepsilon$).
- Magnetic field replacement, induced field $\approx$ applied field. $\mathbf{A} \rightarrow \mathbf{F}$
- Clever choice of gauge to represent the field.
- Mapping to boundary coordinates

$\Rightarrow$ all this previously known, cf Fournais-Helffer's book

Model problem in scaled boundary coordinates, gives the original energy in units of ε^{-1}:

$$\mathcal{E}_{\mathrm{hp}}[\psi] = \int_{s=0}^{|\partial\Omega|\varepsilon^{-1}} \int_{t=0}^{+\infty} \left\{ \left|(\nabla - it\mathbf{e}_s)\,\psi\right|^2 + \frac{1}{b}|\psi|^4 - \frac{2}{b}|\psi|^2 \right\}.$$

- $s =$ tangential coordinate, impose periodicity of ψ in s
- $t =$ normal coordinate
- Only large parameter: length of the domain in s-direction

The boundary problem

- Insert (formally) the ansatz $\psi(s,t) = f(t)e^{-i\alpha s}$
$$\mathcal{E}^{\mathrm{1D}}_{0,\alpha}[f] := \int_0^{+\infty} |\partial_t f|^2 + (t+\alpha)^2 f^2 + \frac{1}{2b}\left(f^4 - 2f^2\right)$$
- Minimize in f and $\alpha \rightsquigarrow$ energy E^{1D}_0, phase α_0, density f_0

Proposition

Let E_{hp} be the infimum of $\mathcal{E}_{\mathrm{hp}}$ under perdiodic boundary conditions in the s-direction. Assume $1 \leq b < \Theta_0^{-1}$, then
$$\frac{|\partial\Omega|}{\varepsilon} E^{\mathrm{1D}}_0 + \mathcal{O}(\varepsilon|\log\varepsilon|) \geq E_{\mathrm{hp}} \geq \frac{|\partial\Omega|}{\varepsilon} E^{\mathrm{1D}}_0.$$

- Trivial upper bound, take trial state of the form
$$\psi(s,t) = f_0(t)\exp\left(-i\varepsilon\left\lfloor\frac{\alpha_0}{\varepsilon}\right\rfloor s\right)$$
- Lower bound is the main part.
- For a lower bound, think of the case where only $|\psi|$ is periodic.

Sketch of the lower bound 1

Inspired by earlier works (Correggi-Pinsker-NR-Yngvason) on the Gross-Pitaevskii theory of rotating superfluids (cf book by Aftalion).

1. State decoupling : since $f_0 > 0$, to any ψ associate a v by setting

$$\psi(s,t) = f_0(t) e^{-i\alpha_0 s} v(s,t).$$

2. Energy decoupling: Variational equation for $f_0 \Rightarrow$ reduced energy

$$\mathcal{E}_{\mathrm{hp}}[\psi] = \frac{|\partial\Omega|}{\varepsilon} E_0^{\mathrm{1D}} + \mathcal{E}_0[v],$$

$$\mathcal{E}_0[v] = \int_{s=0}^{|\partial\Omega|\varepsilon^{-1}} \int_{t=0}^{+\infty} f_0^2(t) \left\{ |\nabla v|^2 - 2(t+\alpha_0)\mathbf{e}_s \cdot \mathbf{j}(v) + \frac{1}{2b} f_0^2(t) \left(1 - |v|^2\right)^2 \right\},$$

with the *superconducting current*

$$\mathbf{j}(v) = \tfrac{i}{2}\left(v \nabla v^* - v^* \nabla v\right) = \rho \nabla \phi \text{ if } v = \sqrt{\rho} e^{i\phi}$$

3. Suffices to prove that the reduced energy is positive for any v

$$\mathcal{E}_0[v] \geq 0.$$

Sketch of the lower bound 2

4. Write $2(t+\alpha_0)f_0^2(t)\mathbf{e}_s = \nabla^\perp F_0$ with a potential function

$$F_0(t,s) = F_0(t) = 2\int_0^t \mathrm{d}\eta\,(\eta+\alpha_0)f_0^2(\eta).$$

5. By definition $F_0 \le 0$, $F_0(0) = F_0(+\infty) = 0$.
6. Stokes' formula

$$\mathcal{E}_0[v] := \int_{s=0}^{|\partial\Omega|\varepsilon^{-1}} \int_{t=0}^{+\infty} f_0^2(t)\,|\nabla v|^2 + F_0(t)\mu(v) + \frac{1}{2b} f_0^4(t)\left(1-|v|^2\right)^2,$$

with the *vorticity*

$$\mu(v) = \operatorname{curl}\mathbf{j}(v), \quad |\mu(v)| \le |\nabla v|^2,$$

7. Then, setting $K_0(t) := f_0^2(t) + F_0(t)$

$$\mathcal{E}_0[v] \ge \int_{s=0}^{|\partial\Omega|\varepsilon^{-1}} \int_{t=0}^{+\infty} K_0(t)\,|\nabla v|^2.$$

8. <u>Lemma:</u> the cost function $K_0(t) \ge 0$ for any $t \in \mathbb{R}^+$ and $1 \le b < \Theta_0^{-1}$.

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Motivation

Local density deviations:

- Pan's conjecture $|\Psi^{\mathrm{GL}}|^2 \to C(b) > 0$ on $\partial\Omega$ does not follow from leading order energy considerations.
- Optimal bound $|\nabla|\Psi^{\mathrm{GL}}|| \propto \varepsilon^{-1}$: holes in the density are repaired over a length scale $\mathcal{O}(\varepsilon)$.
- Density terms come multiplied by $\varepsilon^{-2} \Rightarrow$ potential energy cost of a hole $\sim \varepsilon^{-2} \times$ length$^2 = \mathcal{O}(1)$
- Local density deviations are controled by the $\mathcal{O}(1)$ remainder in previous estimates. Normal inclusions are not ruled out yet.

Role of the curvature:

- Known to play a role in corrections to H_{c3}: Helffer-Morame, Fournais-Helffer, Raymond ...
- Superconductivity starts to appear where curvature is maximum.
- Special behavior of domains with corners (infinite curvature): Bonnaillie-Noël with Dauge, Fournais, Martin-Vial.
- For smooth domains, when $1 < b < \Theta_0^{-1}$, curvature appears only at subleading order.

Reintroducing curvature: case of the disc

- Effective functional in boundary coordinates, including corrections due to curvature $s \mapsto k(s)$:

$$\int_{s=0}^{|\partial\Omega|} \int_{t=0}^{c_0|\log\varepsilon|} (1-\varepsilon k(s)t) \Big\{ |\partial_t \psi|^2 + \frac{1}{(1-\varepsilon k(s)t)^2} \left| (\varepsilon\partial_s + i a_\varepsilon(s,t)) \psi \right|^2 + \frac{1}{2b} \left[|\psi|^4 - 2|\psi|^2 \right] \Big\}$$

with

$$a_\varepsilon(s,t) := -t + \tfrac{1}{2}\varepsilon k(s) t^2 + \varepsilon \delta_\varepsilon, \quad \delta_\varepsilon = \mathcal{O}(1)$$

- Easier case: disc sample, constant curvature k.
- Keep the same ansatz $\psi(s,t) = f(t) e^{-i\alpha s}$, obtain ($c_0 = \mathrm{cst}$)

$$\mathcal{E}^{\mathrm{1D}}_{k,\alpha}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\,(1-\varepsilon k t) \Big\{ |\partial_t f|^2 + \frac{(t+\alpha-\frac{1}{2}\varepsilon k t^2)^2}{(1-\varepsilon k t)^2} f^2 + \frac{1}{2b}\left(f^4 - 2f^2\right) \Big\}$$

Refined results in the disc case

Minimize $\mathcal{E}^{\mathrm{1D}}_{k,\alpha}[f] \rightsquigarrow$ energy $E^{\mathrm{1D}}_{\star}(k)$, phase $\alpha(k)$, density f_k.

Theorem (Correggi-NR 13)

Let Ω be a disc of radius $R = k^{-1}$. For any fixed $1 < b < \Theta_0^{-1}$

$$E^{\mathrm{GL}}_{\varepsilon} = \frac{2\pi E^{\mathrm{1D}}_{\star}(k)}{\varepsilon} + \mathcal{O}(\varepsilon |\log \varepsilon|),$$

and

$$\left\| |\Psi^{\mathrm{GL}}|^2 - f_k^2 \left(\tfrac{R-r}{\varepsilon} \right) \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon^{3/2} |\log \varepsilon|^{1/2}).$$

- Does contain the subleading order:

$$E^{\mathrm{1D}}_{\star}(k) = E^{\mathrm{1D}}_0 + \mathcal{O}(\varepsilon), \quad \alpha(k) = \alpha_0 + \mathcal{O}(\varepsilon), \quad f_k = f_0 + \mathcal{O}(\varepsilon).$$

- Method similar as before, second order cost function.
- Significant but technical additional difficulties.

Refined results in the general case

- Associate $E^{\rm 1D}_\star(k(s)), \alpha_{k(s)}, f_{k(s)}$ to smooth curvature $k(s)$
- Approximate locally the boundary by a disc: think of

$$\Psi^{\rm GL}(\mathbf{r}) = \Psi^{\rm GL}(s,t) \approx f_{k(s)}\left(\tfrac{t}{\varepsilon}\right)\exp\left(-i\alpha_{k(s)}\tfrac{s}{\varepsilon}\right)$$

Theorem (Correggi-NR 14)

For any fixed $1 < b < \Theta_0^{-1}$,

$$E^{\rm GL}_\varepsilon = \frac{1}{\varepsilon}\int_0^{|\partial\Omega|} E^{\rm 1D}_\star\left(k(s)\right)\,{\rm d}s + \mathcal{O}(\varepsilon|\log\varepsilon|^\infty).$$

and

$$\left\| |\Psi^{\rm GL}|^2 - f_{k(s)}\left(\tfrac{t}{\varepsilon}\right)^2 \right\|_{L^2(\Omega)} \leq C\varepsilon^{3/2}|\log\varepsilon|^\infty.$$

- Curvature $k(s) \to$ approximate by constants in cells of side length ε
- Use the disc analysis locally in each cell
- Patch things together and control unphysical boundary terms
- Requires a fine analysis of the k-dependence of the model problem

Effect of curvature on surface superconductivity

- It was previously known (Pan, Fournais-Kachmar ...) that

$$\frac{1}{\varepsilon}|\Psi^{\mathrm{GL}}|^4 \mathrm{d}\mathbf{r} \underset{\varepsilon \to 0}{\longrightarrow} C(b)\mathrm{d}s.$$

- $C(b) > 0$ identified by previous theorems, $\mathrm{d}s$ = 1D Lebesgue measure along the boundary.

Superconductivity density is (roughly) uniform along the boundary.

- Corollary of the previous results: estimate of subleading order

$$\frac{1}{\varepsilon}\left(\frac{1}{\varepsilon}|\Psi^{\mathrm{GL}}|^4 \mathrm{d}\mathbf{r} - C(b)\mathrm{d}s\right) \underset{\varepsilon \to 0}{\longrightarrow} C_2(b)k(s)\mathrm{d}s.$$

- $k(s)$ = curvature.
- $C_2(b) > 0$ (not so) explicitely identified.

Superconductivity density is (slightly) larger in regions of larger curvature.

Thank You !