p 進精度

例と応用

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p-adic methods

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My personal (long-term) motivation

Computing (some) moduli spaces of p-adic Galois representations.

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Definition of the precision

Finite-precision p-adics

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While working with a computer, we usually only can consider the beginning of this power serie expansion: we only consider elements of the following form $\boxed{\sum_{i=l}^{d-1} a_i p^i + O(p^d)}$, with $l \in \mathbb{Z}$.

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Example

The order of $3 * 7^{-1} + 4 * 7^0 + 5 * 7^1 + 6 * 7^2 + O(7^3)$ is 3.

p-adic precion *vs* real precision

The quintessential idea of the step-by-step analysis is the following :

Proposition (p-adic errors don't add)

Indeed,

$$(a + O(p^k)) + (b + O(p^k)) = a + b + O(p^k).$$

That is to say, if a and b are known up to precision $O(p^k)$, then so is $a + b$.

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Remark

It is quite the opposite to when dealing with real numbers, because of **Round-off error** :

$$(1 + 5 * 10^{-2}) + (2 + 6 * 10^{-2}) = 3 + 1 * 10^{-1} + 1 * 10^{-2}.$$

That is to say, if a and b are known up to precision 10^{-n}, then $a+b$ is known up to $10^{(-n+1)}$.

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Precision formulae

Proposition (addition)

$$(x_0 + O(p^{k_0})) + (x_1 + O(p^{k_1})) = x_0 + x_1 + O(p^{min(k_0,k_1)})$$

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1} p^{a-c} + O(p^{min(d+a-2c,b-c)})$$

In particular,

$$\frac{1}{p^c y + O(p^d)} = y^{-1} p^{-c} + O(p^{d-2c})$$

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A little warm-up on computing determinants : expansion

An example of determinant computation

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If we expand directly using the expression of the determinant in terms of the coefficients, we get:

$$-2p^9 + O(p^{10}),$$

because of $1 \times 1 \times O(p^{10})$.

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- Depends heavily on the algorithm chosen for the computation
- No idea on what is **optimal**.

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The Main lemma of p-adic differential precision

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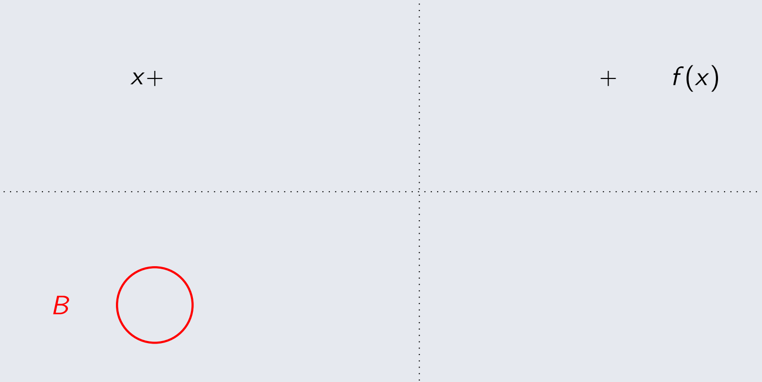
Geometrical meaning

Interpretation



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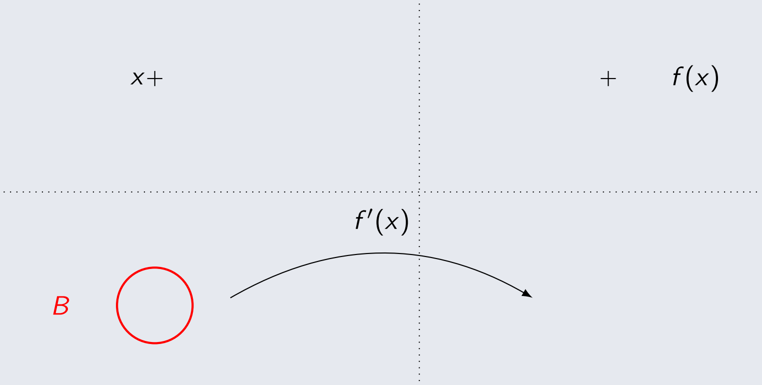

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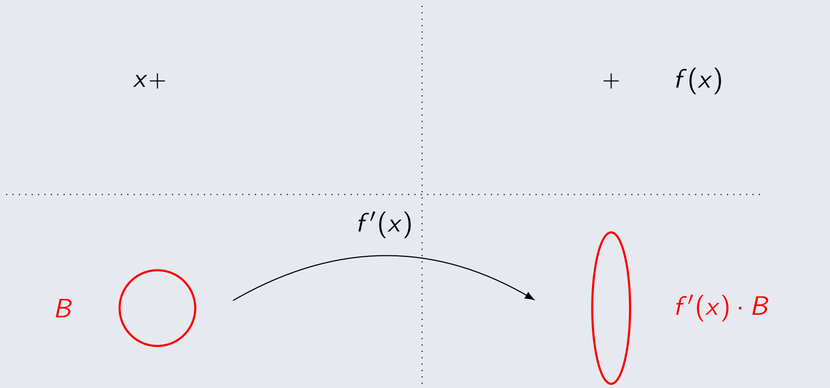

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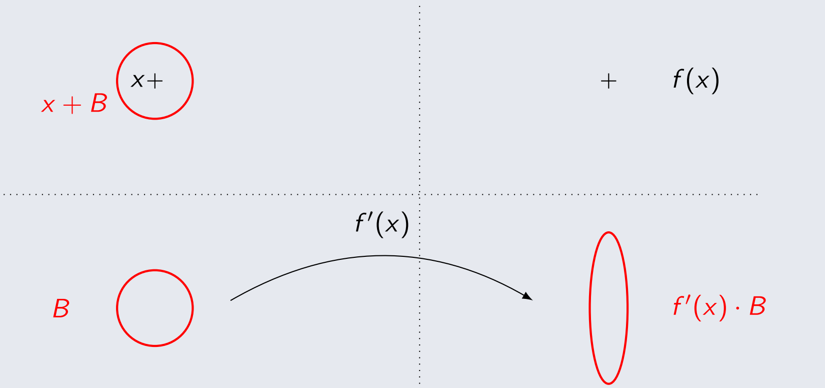

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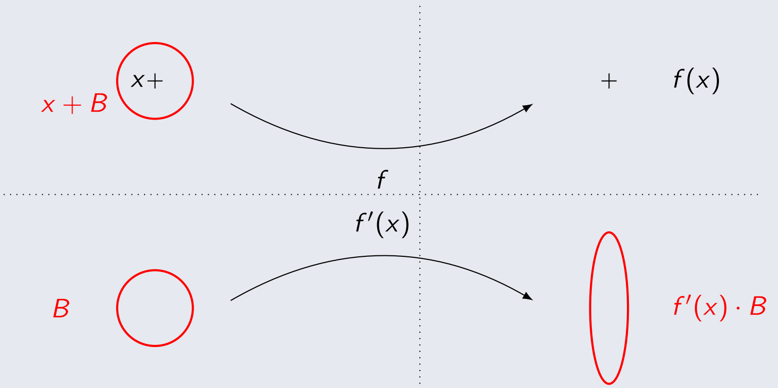

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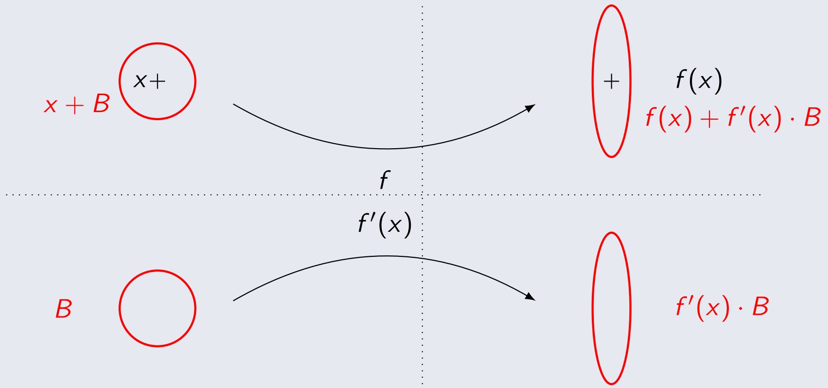

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Remark

Our framework can be extended to **(complete) ultrametric K-vector spaces** (*e.g.* being $\mathbb{F}_p((X))^n$, $\mathbb{Q}((X))^m$, $\mathbb{R}((\varepsilon))^s$).

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This can be determined with **Newton-polygon** techniques.

Looking back to the case of the determinant

Differential of the determinant

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- **Approximate SNF is optimal.**

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Some more examples

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Point-counting algorithms

Use isogenies between an elliptic curve E and other curves: quotient by l-torsion, ...

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Use isogenies between an elliptic curve E and other curves: quotient by l-torsion, ...

Cryptosystems

De Feo-Jao-Plût (2011) have proposed cryptosystems based in the computation of isogenies.

Toward computation

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Isogeny and Differential equations (*cf* Schoof-Elkies-Atkin algorithm, Bostan-Morain-Salvy-Schost 08, Lercier-Sirvent 08, . . .)

Let E and $\tilde{E}$ be two elliptic curves over $\mathbb{Z}/p\mathbb{Z}$:

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Writing $U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}$, we get :

$$(Bx^6 + Ax^4 + 1)S'^2 = 1 + \tilde{A}S^4 + \tilde{B}S^6.$$

Change of variable and the differential equation

The differential equation

Let S be such that

$$U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}.$$

Then if $A, B, \tilde{A}, \tilde{B}$ are in $\mathbb{Z}_p$,

$$S \in \mathbb{Z}_p[[t]]$$

We have the following differential equation for S :

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A p-adic computation of a solution

Computing the isogeny

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3. Reduce mod p to get the solution in $\mathbb{Z}/p\mathbb{Z}$.

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Change of equation

When $p \neq 2$, we can replace $y'^2 \times G = H(y)$ by $y' = g \times h(y)$ with $g, h \in \mathbb{Z}_p[[x]]$, $g(0) = h(0) = 1$

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To compute $y \mod x^{2^N+1}$, we need an initial precision of $O(N^2)$ digits.

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Differential and differential equation

Theorem

Let $\Phi : (g, h) \mapsto y$ *such that* $y(0) = 0$ *and* $y' = gh(y)$*. Then,*

$$\Phi'(g, h) \cdot (\delta g, \delta h) = h(y) \int \delta g + \frac{g \delta h(y)}{h(y)}.$$

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$$\Phi'(y)\cdot(\delta g, \delta h) \mod x^{2^N+1} \in \frac{O(p^k)}{p^N}\mathbb{Z}_p[\![x]\!].$$

First conclusion on the application of the lemma

Proposition

$\Phi(g,h) \mod (p, t^{2^n})$ *is determined by* $g, h \mod (p^{1+\log_p 2^n}, t^{2^n}))$*. In other words, we have a logarithmic loss in precision.*

What happens in practice ?

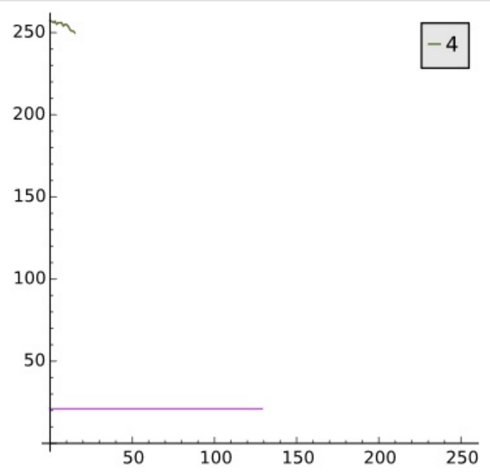

Figure: Precision over the output

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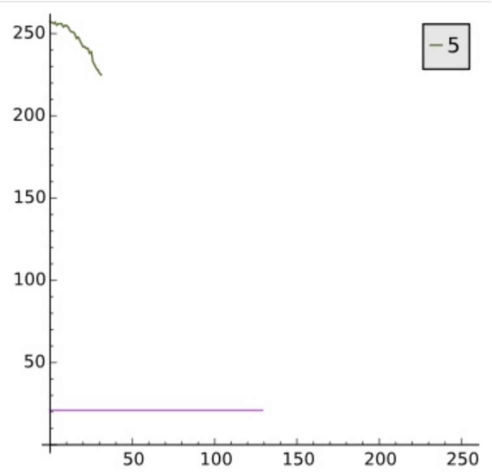

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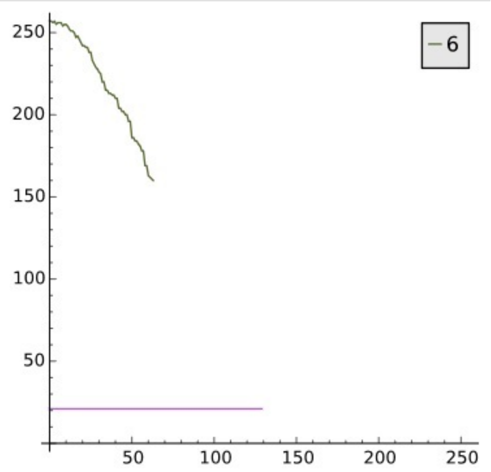

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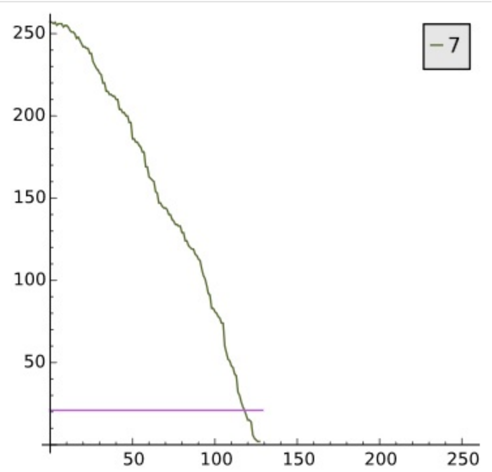

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Another take on the computation

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- Another way is then to modify the current g, h, u_0 **at each step**, in a consistent way, so as to keep on getting better approximate solutions.
- A third way here will be to work entirely in $\mathbb{Z}/p^\kappa\mathbb{Z}$.

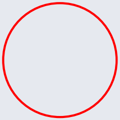
Adaptative method

Adaptative differential tracking of precision

$x + O(p^N)$

$x+$

B



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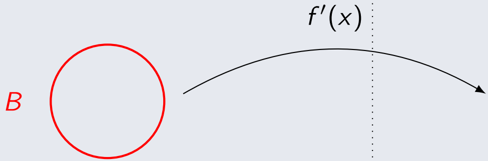

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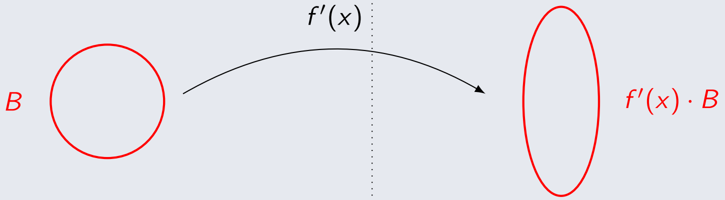

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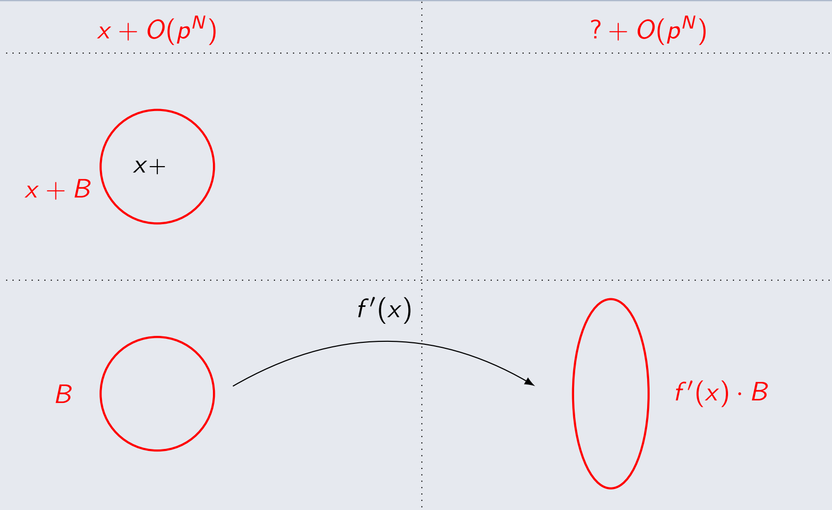

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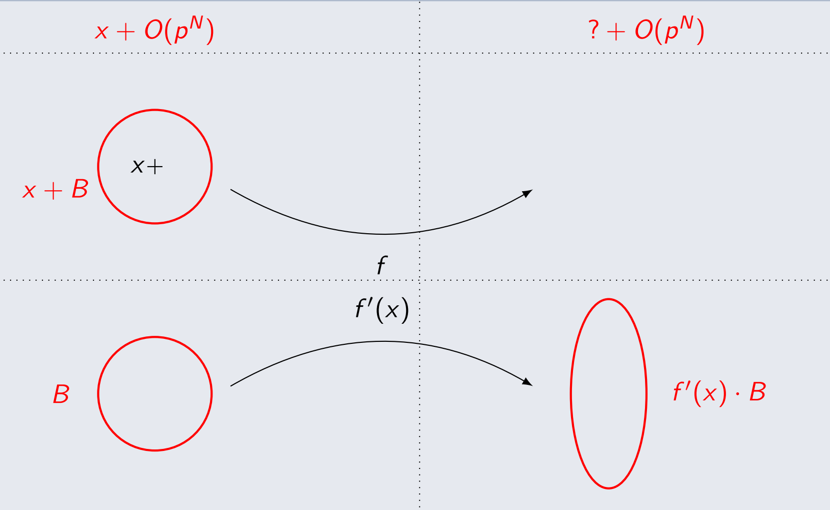

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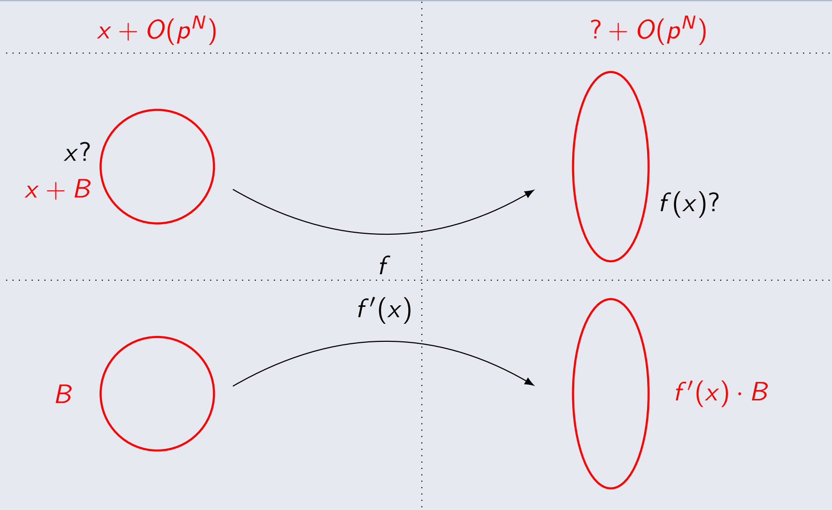

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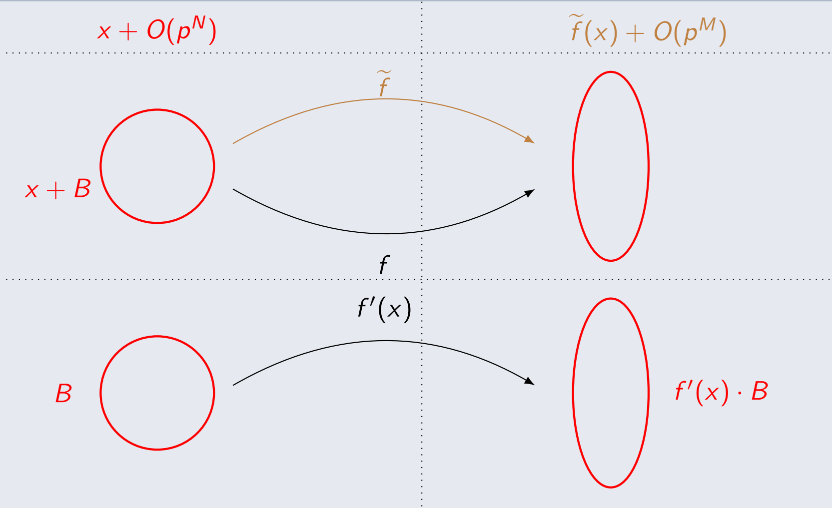

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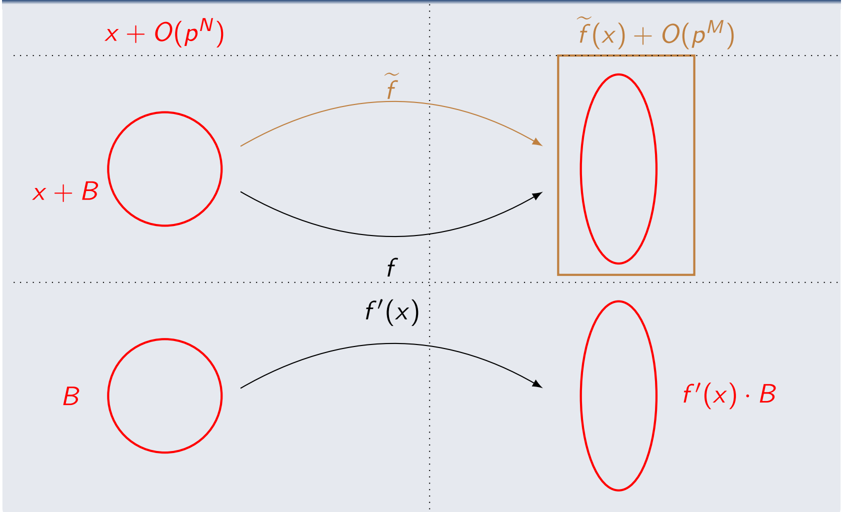

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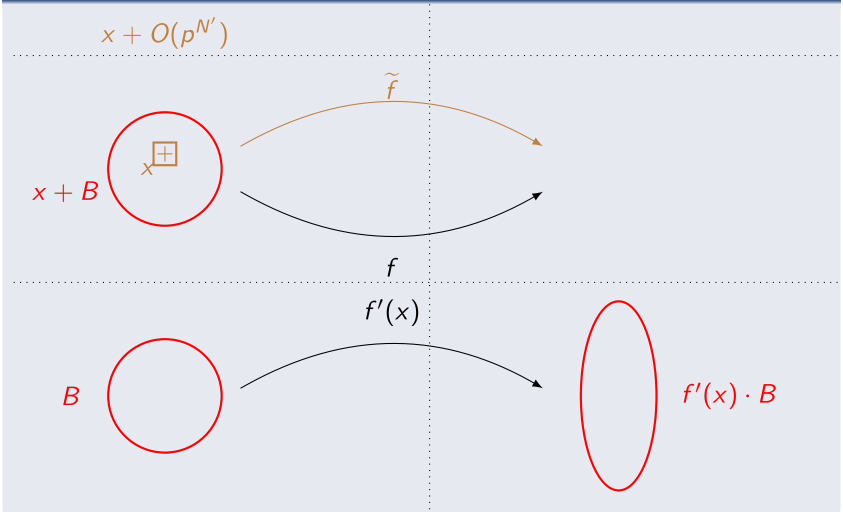

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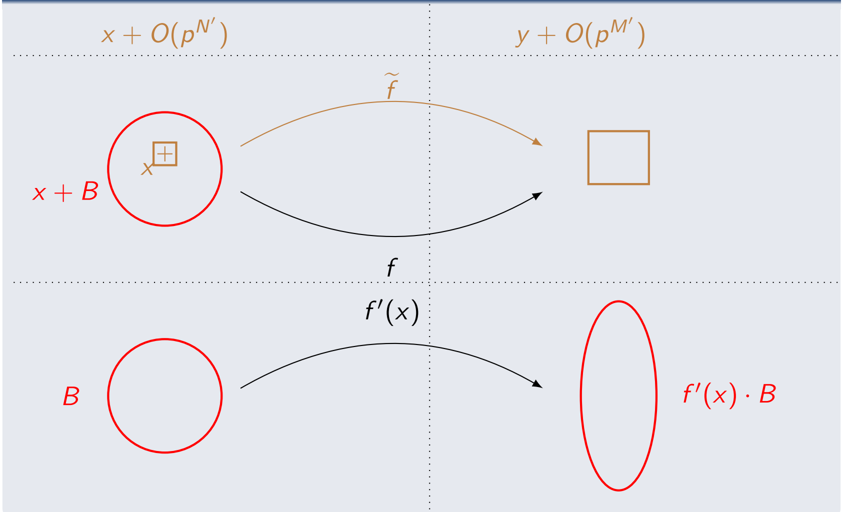

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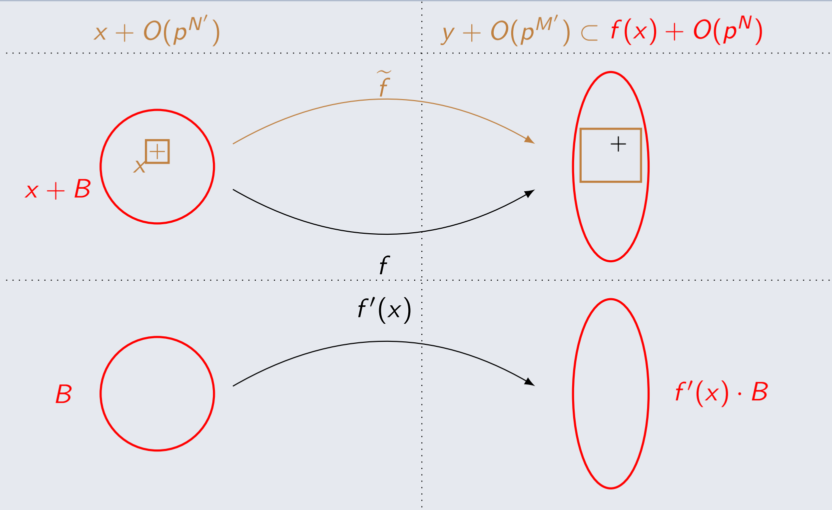

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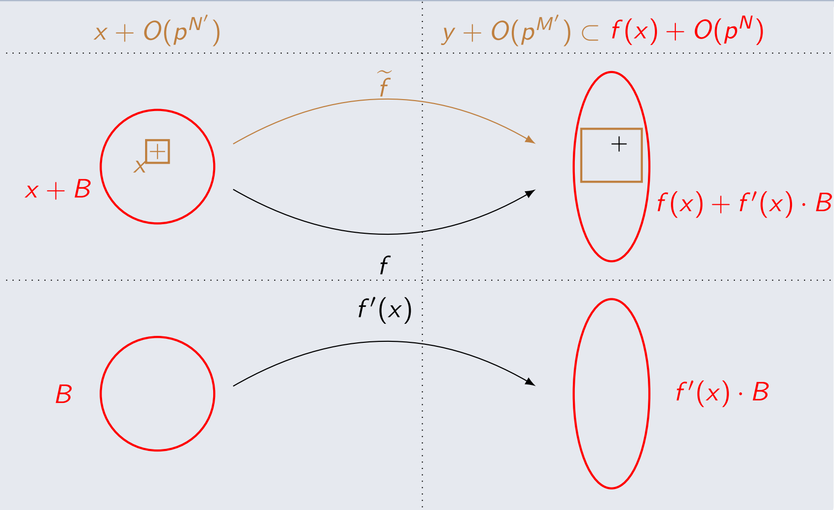

Final take on the Newton scheme

We can prove that it is harmless to work in $\mathbb{Z}/p^k\mathbb{Z}$ for our computation.

Proposition

We can obtain the solution $\Phi(g,h) \mod (p, t^{n+1})$ *knowing* $g, h \mod (p^{\lfloor \log_p n \rfloor + 1}, t^{n+1})$ *and applying the following iteration:*

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modulo $p^{\lfloor \log_p n \rfloor + 1}$ *and growing order of truncation.*

Timings

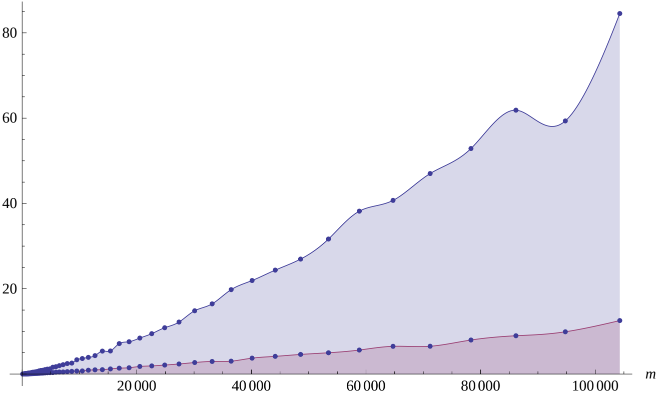

Figure: Timings in seconds, measured on a laptop, of our Algorithm run at precision λ_{old} (upper curve) and λ_{new} (lower curve) in order to compute an approximation modulo $(5, t^{4m+1})$ of some given m-isogenies.

Speedup

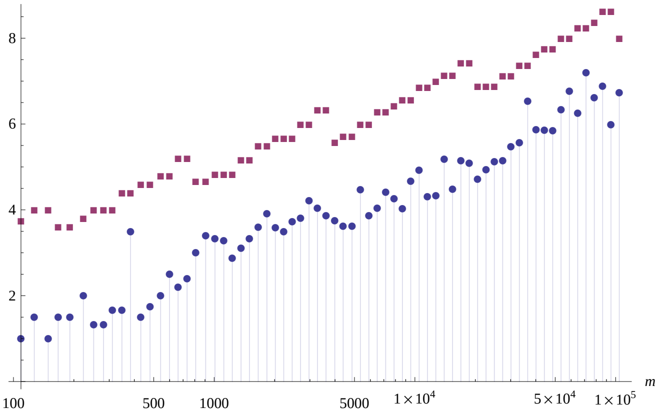

Figure: Practical speedup obtained with the new precision analysis compared with the theoretical improvement (m-axis in logarithmic scale). (■) is the ratio on precisions, (●) is the actual speedup.

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What happens?

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What happens when $p = 2$? Square roots are very costly in $\mathbb{Q}_2$.

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-n+1)}{n!}x^n + o(x^n).$$

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g, h are even, S is odd.

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$$\phi^{-1} : v \mapsto v'I - \frac{1}{2}vI'.$$

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References

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Linear Algebra

- Xavier Caruso, David Roe and Tristan Vaccon p-adic stability in linear algebra, ISSAC 2015.

Differential equations

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Thank you for your attention

Thanks