NODAL DEFICIENCY OF RANDOM SPHERICAL HARMONICS IN PRESENCE OF BOUNDARY

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Random Nodal Domains
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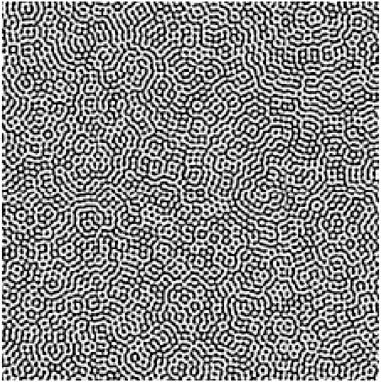
Boundary-adapted random wave model

Berry's Random Wave Model



Random monochromatic waves [Berry 1977]: centred isotropic Gaussian random field $u : \mathbb{R}^2 \to \mathbb{R}$ described by the covariance function

$$\mathbb{E}[u(x) \cdot u(y)] = J_0(||x - y||), \qquad x, y \in \mathbb{R}^2.$$

Berry's Random Wave Model

Zero density (in this isotropic case):

$$K_1^u(x) = \phi_{u(x)}(0) \cdot \mathbb{E}[\|\nabla u(x)\| \big| u(x) = 0] \equiv \frac{1}{2\sqrt{2}}.$$

Nodal length restricted to a radius-R disc $\mathcal{B}(R)$: $\mathcal{L}(u;R)$.
Expected nodal length (Kac-Rice formula):

$$\mathbb{E}[\mathcal{L}(u;R)] = \int\limits_{\mathcal{B}(R)} K_1^u(x)dx = \frac{1}{2\sqrt{2}} \cdot \mathrm{Area}(\mathcal{B}(R)).$$

Asymptotic law for the variance [Berry 2002] (Kac-Rice formula):

$$\mathrm{Var}(\mathcal{L}(u;R)) = \frac{1}{256} \cdot R^2 \log R + O(R^2).$$

smaller than the heuristic prediction $\mathrm{Var}(\mathcal{L}(u;R)) \approx R^3$, "Berry's cancellation" of the leading non-oscillatory term of the 2-point correlation function.

Boundary-adapted random wave model

[Berry 2002] studied the effect of the Dirichlet condition on a boundary $\{(x_1, x_2) :\ x_2 = 0\} \subseteq \mathbb{R}^2$ both in its vicinity and far away from it.

Boundary-adapted (non-stationary) random waves: $v : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ is the limit, as $J \to \infty$, of the superposition

$$\frac{2}{\sqrt{J}} \sum_{j=1}^{J} \sin(x_2 \sin(\theta_j)) \cdot \cos(x_1 \cos(\theta_j) + \phi_j)$$

of J plane waves forced to vanish at $x_2 = 0$.

v is the centred Gaussian field with covariance function

$$r_v(x, y) := \mathbb{E}[v(x) \cdot v(y)] = J_0(\|x - y\|) - J_0(\|x - \widetilde{y}\|),$$

$x = (x_1, x_2)$, $y = (y_1, y_2)$, $\widetilde{y} = (y_1, -y_2)$ (mirror symmetry relatively to x axis).

The law is invariant w.r.t. horizontal shifts $v(\cdot, \cdot) \mapsto v(a + \cdot, \cdot)$, $a \in \mathbb{R}$, but not the vertical shifts.

$$r_u(x,y) = J_0(||x-y||), \qquad r_v(x,y) = J_0(\|x-y\|) - J_0(\|x-\widetilde{y}\|),$$

- in the range $x_2, y_2 \to \infty$, $r_v(x,y) \approx J_0(\|x-y\|)$ at infinity the boundary has a small impact
- the decay of the error term is slow and of oscillatory nature
- it takes its toll on the *nodal bias*

as $x_2 \to 0$, the zero density of v, depending on the height x_2 and independent of x_1 by the invariance, is s.t.

$$K_1^v(x) = K_1^v(x_2) \to \frac{1}{2\pi} < \frac{1}{2\sqrt{2}} = K_1^u(x).$$

Berry attributed this *nodal deficiency* to the a.s. orthogonality of the nodal lines touching the boundary [Cheng, 1976].

- As $x_2 \to \infty$,

$$K_1^v(x_2) = \frac{1}{2\sqrt{2}} \cdot \left(1 + \frac{\cos(2x_2 - \pi/4)}{\sqrt{\pi x_2}} - \frac{1}{32\pi x_2} + E(x_2)\right).$$

A natural choice for expanding domains are the rectangles

$$\mathcal{D}_R := [-1,1] \times [0,R], \qquad R \to \infty,$$

Kac-Rice formula:

$$\mathbb{E}[\mathcal{L}(v;\mathcal{D}_R)] = \frac{1}{2\sqrt{2}} \cdot \mathrm{Area}(\mathcal{D}_R) - \frac{1}{32\sqrt{2}\pi} \log R + O(1)$$

logarithmic *nodal deficiency*.

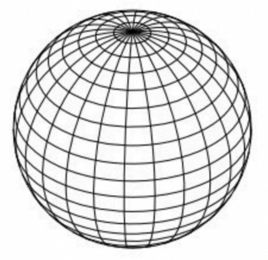
Local effect of the perpendicular intersection of the nodal line with the boundary $+$ negative excess term infinitely many wavelengths away from the boundary.

The logarithmic fluctuations in the isotropic case u, *possibly* also holding for v, give rise to a hope to be able to detect the logarithmic negative boundary impact via a single sample of the nodal length or very few ones.

Boundary-adapted random spherical harmonics

Random spherical harmonics

$\mathcal{S}^2$ (unit) sphere, Laplace eigenvalues are all the numbers $\ell(\ell+1)$, for ℓ nonnegative integer. The corresponding eigenspace, is $2\ell+1$-dim. space of spherical harmonics of degree ℓ, let $\mathcal{E}_\ell := \{\eta_{\ell,1}, \dots \eta_{\ell,2\ell+1}\}$ be its arbitrary L^2-orthonormal basis.



Degree-ℓ random spherical harmonics

$$\widetilde{T}_\ell(x) = \sqrt{\frac{4\pi}{2\ell+1}} \sum_{k=1}^{2\ell+1} a_k \cdot \eta_{\ell,k}(x), \qquad x \in \mathcal{S}^2,$$

a_k i.i.d. standard Gaussian random variables.

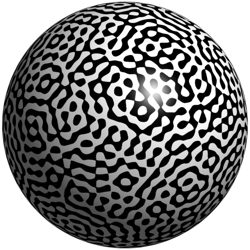
Law of $\widetilde{T}_\ell$ invariant w.r.t. the chosen orthonormal basis $\mathcal{E}_\ell$, uniquely defined via the covariance function

$$\mathbb{E}[\widetilde{T}_\ell(x) \cdot \widetilde{T}_\ell(y)] = P_\ell(\cos d(x,y)),$$

$d(\cdot,\cdot)$ is the spherical distance between $x, y \in \mathcal{S}^2$ (isotropy).

[Berard, 1985] evaluated the expected total nodal length

$$\mathbb{E}[\mathcal{L}(\widetilde{T}_\ell)] = \sqrt{2\pi} \cdot \sqrt{\ell(\ell+1)}.$$



As $\ell \to \infty$ its variance is asymptotic to [Wigman, 2010]

$$\mathrm{Var}(\mathcal{L}(\widetilde{T}_\ell)) \sim \frac{1}{32} \log \ell,$$

in accordance with random monochromatic waves [Berry 2002], as $R \to \infty$

$$\mathrm{Var}(\mathcal{L}(u;R)) = \frac{1}{256} \cdot R^2 \log R + O(R^2),$$

save for the scaling, and the invariance of the nodal lines w.r.t. the symmetry $x \mapsto -x$ of the sphere, resulting in a doubled leading constant suitably scaled.

Boundary-adapted random spherical harmonics

$\mathcal{H}^2 \subseteq \mathcal{S}^2$ hemisphere with *Dirichlet* boundary conditions along the equator.

[Hassell-Tao, 2002 Example 4] the eigenfunctions are given by those spherical harmonics which are odd under reflection in the (x_1, x_2) plane, namely, spherical harmonics where $-\ell \leq m \leq \ell$ and $\ell - m$ is odd.

For $\ell \geq 0$, $|m| \leq \ell$ the spherical harmonic $Y_{\ell,m}$ obeys the Dirichlet boundary condition on the equator, if and only if $m \not\equiv \ell \mod 2$, spanning a subspace of dimension ℓ inside the $(2\ell + 1)$-dimensional space of spherical harmonics of degree ℓ.

(Its $(\ell + 1)$-dimensional orthogonal complement is the subspace satisfying the Neumann boundary condition).

Boundary-adapted random spherical harmonics

$$T_\ell(x) = \sqrt{\frac{8\pi}{2\ell+1}} \sum_{\substack{m=-\ell \\ m \not\equiv \ell \mod 2}}^{\ell} a_{\ell,m} Y_{\ell,m}(x),$$

$a_{\ell,m}$ standard complex-valued Gaussian s.t. $a_{\ell,-m} = \overline{a_{\ell,m}}$, T_ℓ real-valued, with *covariance function*:

$$\mathbb{E}[T_\ell(x) \cdot T_\ell(y)] = P_\ell(\cos d(x,y)) - P_\ell(\cos d(x,\overline{y})),$$

where $\overline{y}$ is the mirror symmetry of y around the equator

$$y = (\theta, \phi) \mapsto \overline{y} = (\pi - \theta, \phi).$$

Proof: use $Y_{\ell,m}(\theta,\phi) = (-1)^{\ell+m} Y_{\ell,m}(\pi-\theta,\phi)$ and Addition Theorem.

The law of T_ℓ is invariant (see definition or covariance) w.r.t. rotations of $\mathcal{H}^2$ around the axis orthogonal to the equator, in the spherical coordinates

$$T_\ell(\theta,\phi) \mapsto T_\ell(\theta, \phi+\phi_0), \quad \phi \in [0, 2\pi)$$

The mirror symmetry $y \mapsto \widetilde{y}$ relatively to the x axis in the Euclidean situation is substituted by mirror symmetry $y \mapsto \overline{y}$ relatively to the equator.

Zero density

$$K_{1,\ell}(x) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{\mathrm{Var}(T_\ell(x))}} \mathbb{E}\big[\|\nabla T_\ell(x)\| \big| T_\ell(x) = 0\big],$$

unlike the rotation invariant spherical harmonics it genuinely depends on $x \in \mathcal{H}^2$. More precisely, the zero density $K_{1,\ell}(x)$ depends on the polar angle θ only.

We rescale by introducing the variable

$$\psi = \ell(\pi - 2\theta),$$

and, with a slight abuse of notation, write

$$K_{1,\ell}(\psi) = K_{1,\ell}(x).$$

Our principal result is on the asymptotics of $K_{1,\ell}$ in two different regimes.

Theorem (C.-Marinucci-Wigman, 2021)

1. *For $C > 0$ sufficiently large, as $\ell \to \infty$, one has*

$$K_{1,\ell}(\psi) = \frac{\sqrt{\ell(\ell+1)}}{2\sqrt{2}} \left[1 + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\psi}} \cos\{(\ell+1/2)\psi/\ell - \pi/4\} - \frac{1}{16\pi\psi} \right.$$
$$\left. + \frac{15}{16\pi\psi} \cos\{(\ell+1/2)2\psi/\ell - \pi/2\} \right] + O(\psi^{-3/2}\ell^{-2}),$$

uniformly for $C < \psi < \pi\ell$, with the constant involved in the 'O'-notation absolute.

2. *For $\ell \geq 1$ one has the uniform asymptotics*

$$K_{1,\ell}(\psi) = \frac{\ell}{2\pi} \left[1 + O(\ell^{-1}) + O(\psi^2) \right],$$

with the constant involved in the 'O'-notation absolute. (This is asymptotic for ψ small only, otherwise yielding the mere bound $K_{1,\ell}(\psi) = O(\ell)$.)

Proof

$$K_{1,\ell}(x) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{\mathrm{Var}(T_\ell(x))}} \mathbb{E}\big[\|\nabla T_\ell(x)\| \big| T_\ell(x) = 0\big]$$

- we naturally encounter the distribution of $T_\ell(x)$, determined by

$$\mathrm{Var}(T_\ell(x)) = 1 - P_\ell(\cos d(x, \bar{x})),$$

- and the distribution of $\nabla T_\ell(x)$ conditioned on $T_\ell(x) = 0$, is determined by its 2×2 covariance matrix

$$\mathbf{\Omega}_\ell(x) = \mathbb{E}[\nabla T_\ell(x) \cdot \nabla^t T_\ell(x) | T_\ell(x) = 0] = \frac{\ell(\ell+1)}{2} \left[\mathbf{I}_2 + \mathbf{S}_\ell(x)\right]$$

depending only on θ (explicit computation).

Perturbative analysis away from the boundary

- evaluate the variance $\mathrm{Var}(T_\ell(x))$ and each entry in $\mathbf{S}_\ell(x)$ using the high degree asymptotics of the Legendre polynomials and its derivatives (Hilb's asymptotics).
- exploit the analyticity of the Gaussian expectation $K_{1,\ell}$ as a function of the entries of the corresponding *non-singular* covariance matrix, to Taylor expand $K_{1,\ell}(x)$ where both $\mathrm{Var}(T_\ell(x)) - 1$ and the entries of $\mathbf{S}_\ell(x)$ are small.

Perturbative analysis at the boundary

- study the asymptotic behaviour of the density function $K_{1,\ell}(\psi)$ for $0 < \psi < \epsilon_0$ with $\epsilon_0 > 0$ sufficiently small.

Kac-Rice formula

Theorem (C.-Marinucci-Wigman, 2021)

The expected nodal length of T_ℓ satisfies

$$\mathbb{E}[\mathcal{L}(T_\ell)] = \int_{\mathcal{H}^2} K_{1,\ell}(x)dx + 2\pi,$$

where $K_{1,\ell}(\cdot)$ is the zero density of T_ℓ.

- Kac-Rice formula outside the equator is verified via an explicit computation (non-degeneracy of the covariance matrix at all these points)
- the non-degeneracy conditions fail at the equator $\mathcal{E} = \{(\theta, \phi) :\ \theta = \pi/2\} \subseteq \mathcal{H}^2$
- we excise a small neighbourhood of this degenerate set, and apply the Monotone Convergence Theorem so to be able to prove that Kac-Rice holds precisely, save for the length of the equator
- the equator is bound to be contained in the nodal set of T_ℓ, by the Dirichlet boundary condition.

Expected nodal length

As a corollary, one may evaluate the asymptotic law of the total expected nodal length of T_ℓ, and detect the negative logarithmic bias relatively to [Berard, 1985]

$$\mathbb{E}[\mathcal{L}(\widetilde{T}_\ell)] = \sqrt{2\pi} \cdot \sqrt{\ell(\ell+1)},$$

in full accordance with [Berry, 2002].

Corollary (C.-Marinucci-Wigman, 2021)

As $\ell \to \infty$, the expected nodal length has the following asymptotics:

$$\mathbb{E}[\mathcal{L}(T_\ell)] = 2\pi \frac{\sqrt{\ell(\ell+1)}}{2\sqrt{2}} - \frac{1}{32\sqrt{2}} \log(\ell) + O(1).$$

Proof

We separate the contribution of the following three subregions of the hemisphere $\mathcal{H}^2$ in the Kac-Rice integral:

$$\mathcal{H}_F = \{(\psi, \phi) : C < \psi < \pi\ell\}$$

$$\mathcal{H}_C = \{(\psi, \phi) : 0 < \psi < \epsilon_0\}$$

$$\mathcal{H}_I = \{(\psi, \phi) : \epsilon_0 < \psi < C\}$$

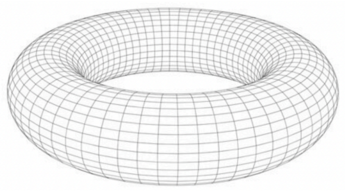
- $\mathcal{H}_F$ gives the main contribution
- the contribution of $\mathcal{H}_C$ is bounded recalling the uniform estimate of $K_{1,\ell}$
- intermediate range
  - the variance at the denominator is bounded away from 0
  - the diagonal entries of the unconditional covariance matrix are $O(\ell^2)$ and the diagonal entries of the conditional matrix are bounded by the unconditional ones from Gaussian Correlation Inequality [Royen, 2014]

$$\begin{aligned}\mathbb{E}[\|\nabla T_\ell(\psi/\ell)\| \big| T_\ell(\psi/\ell) = 0] &\leq \left(\mathbb{E}[\|\nabla T_\ell(\psi/\ell)\|^2 \big| T_\ell(\psi/\ell) = 0]\right)^{1/2} \\ &\leq \left(\mathbb{E}[\|\nabla T_\ell(\psi/\ell)\|^2]\right)^{1/2} = O(\ell).\end{aligned}$$

Square with Dirichlet boundary

Arithmetic Random Waves

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, Laplace eigenvalues are all the number $4\pi^2 n$, n integer expressible as a sum of two squares, corresponding eigenspace is the collection of all (complex) linear combinations of the plane waves



$$e^{2\pi i \langle \mu, x \rangle}$$

$\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$ lattice points lying on the radius-$\sqrt{n}$ centred circle.

Arithmetic Random Waves [Oravecz-Rudnick-Wigman, 2008] is the Gaussian ensemble $f_n : \mathbb{T}^2 \to \mathbb{R}$

$$f_n(x) = \sum_{\|\mu\|^2 = n} a_\mu \, e^{2\pi i \langle \mu, x \rangle},$$

a_μ standard complex i.i.d. Gaussian, save for $a_{-\mu} = \overline{a_\mu}$ making f_n real.

Expected nodal length [Rudnick-Wigman, 2008] $\mathbb{E}[\mathcal{L}(f_n)] = \sqrt{2}\pi^2 \cdot \sqrt{n}$.

Square with Dirichlet boundary

[C.-Klurman-Wigman, 2020] compares the torus to the square with Dirichlet boundary.

The total nodal bias fluctuates from nodal deficiency (negative bias) to nodal surplus (positive bias), depending on the angular distribution of the lattice points and its interaction with the direction of the square boundary, at least, for generic energy levels.

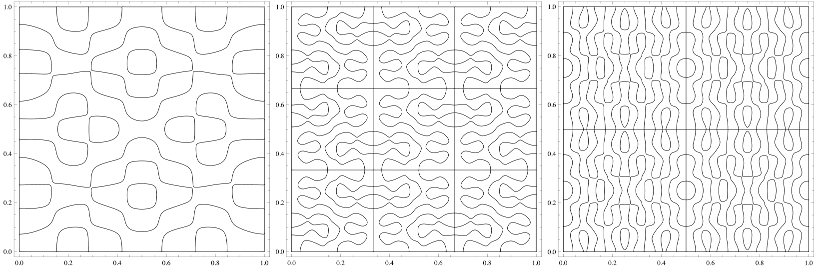


Figure: Nodal line for $n = 170$, $n = 765$, $n = 1000$.

The degenerate set consists of a union of a grid and finitely many isolated point, by Monotone Convergence Theorem, it is possible to deduce that Kac-Rice holds precisely, save for the length of the deterministic grid contained in the nodal set.