On the number of roots of Sturm-Liouville random sums

Joint work with José R. León.

Federico Dalmao

DEPARTAMENTO DE MATEMÁTICA Y ESTADÍSTICA DEL LITORAL
CENUR Litoral Norte
UNIVERSIDAD DE LA REPÚBLICA URUGUAY

Random Nodal Domains
Rennes, June 6th 2023

Plan:

1. Introduction.
   1.1 About Random trigonometric polynomials.
   1.2 A few generalizations.
   1.3 Motivation.
2. Some preliminaries.
3. Problem setting and main results.
4. Sketch of the proof.

Introduction

Introduction - About Random trigonometric polynomials.

There exist two main variants of RTP.

Qualls

$$T_n(x) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx).$$

Classical

$$C_n(x) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \cos(kx).$$

The first works date from the 60s and deal with the asymptotic behavior of the expectation in the case of standard Gaussian coefficients. Das, Dunnage, Wilkins proved that the number of roots N_n of both T_n and C_n on $[0, 2\pi]$ verifies

$$\mathbf{E}\left(N_n\right) \underset{n\to\infty}{\sim} \frac{2n}{\sqrt{3}}.$$

Introduction - About Random trigonometric polynomials.

The behavior of this ensemble of random polynomials (and of its generalizations regarding the law of the coefficients) is nowadays well understood since it has been intensively studied by Granville, Wigman, Angst, Pautrel, Poly, Coutin, Peralta, Flasche, etc.

1. The asymptotic expectation is universal in the iiid case.
2. The asymptotic variance is not universal in the iiid case.
3. In the correlated Gaussian case a whole interval of possible limit expectations appears, or even we can observe an oscilation.
4. In the iid Gaussian case, the variance grows linearly with n and a CLT holds true.

Introduction - A few generalizations.

Today, we want to generalize in a particular direction this setting. The idea is to replace sines and cosines by another class of convenient/interesting functions.

1. Angst and Poly studied periodic signals.
2. Do, Nguyen, Nguyen and Pritsker studied the number of (real) roots of orthogonal polynomials defined on curves.

Introduction - Motivation.

This work was somehow motivated by the paper

Do, Yen; Nguyen, Hoi H; Nguyen, Oanh; Pritsker, Igor. Central limit theorem for the number of real roots of random orthogonal polynomials. arxiv.org/abs/2111.09015.

(Here, we do not have stationarity, the result is obtained via chaotic expansions.)

It remembered us our first works on random polynomials, about the number of roots of Qualls' and classical random trigonometric polynomials..

This work was somehow motivated by the paper

Do, Yen; Nguyen, Hoi H; Nguyen, Oanh; Pritsker, Igor. Central limit theorem for the number of real roots of random orthogonal polynomials. arxiv.org/abs/2111.09015.

(Here, we do not have stationarity, the result is obtained via chaotic expansions.)

It remembered us our first works on random polynomials, about the number of roots of Qualls' and classical random trigonometric polynomials..

Introduction - Motivation.

A trigonometric polynomial can be thought as a finite Fourier expansion of a given function. Thus, random trigonometric polynomials are, in this sense, finite Fourier expansions of random functions, that is, expansions in the orthogonal basis formed by sines and cosines.

Naturally, one can think of other similar expansions as that in terms of Bessel functions.

Fourier and Bessel expansions are particular cases of Sturm-Liouville expansions.

Let $q : [a, b] \to \mathbb{R}$ be a positive (or bounded by below) and continuous function with finite limits at a, b and consider the differential operator

$$\mathcal{L} := q(x) - \frac{d^2}{dx^2}, \tag{1}$$

which acts on smooth functions defined on $[a, b]$. Under these conditions on q, the problem is called regular.

When $q = 0$ we retrieve the trigonometric/Fourier case while if $q(x) = (\nu - \frac{1}{4})\frac{1}{x^2}$ we recover the Bessel case.

Introduction - Motivation.

A trigonometric polynomial can be thought as a finite Fourier expansion of a given function. Thus, random trigonometric polynomials are, in this sense, finite Fourier expansions of random functions, that is, expansions in the orthogonal basis formed by sines and cosines.

Naturally, one can think of other similar expansions as that in terms of Bessel functions.

Fourier and Bessel expansions are particular cases of Sturm-Liouville expansions.

Let $q : [a, b] \to \mathbb{R}$ be a positive (or bounded by below) and continuous function with finite limits at a, b and consider the differential operator

$$\mathcal{L} := q(x) - \frac{d^2}{dx^2}, \tag{1}$$

which acts on smooth functions defined on $[a, b]$. Under these conditions on q, the problem is called regular.

When $q = 0$ we retrieve the trigonometric/Fourier case while if $q(x) = (\nu - \frac{1}{4})\frac{1}{x^2}$ we recover the Bessel case.

Introduction - Motivation.

A trigonometric polynomial can be thought as a finite Fourier expansion of a given function. Thus, random trigonometric polynomials are, in this sense, finite Fourier expansions of random functions, that is, expansions in the orthogonal basis formed by sines and cosines.

Naturally, one can think of other similar expansions as that in terms of Bessel functions.

Fourier and Bessel expansions are particular cases of Sturm-Liouville expansions.

Let $q : [a, b] \to \mathbb{R}$ be a positive (or bounded by below) and continuous function with finite limits at a, b and consider the differential operator

$$\mathcal{L} := q(x) - \frac{d^2}{dx^2}, \tag{1}$$

which acts on smooth functions defined on $[a, b]$. Under these conditions on q, the problem is called regular.

When $q = 0$ we retrieve the trigonometric/Fourier case while if $q(x) = (\nu - \frac{1}{4})\frac{1}{x^2}$ we recover the Bessel case.

The classical theory, states that under mild conditions, an integrable function f can be expanded in terms of the eigenfunctions of $\mathcal{L}$, being the behavior of this expansion similar to that of the Fourier case.

Introduction - Motivation.

Fourier expansions can be used in the solution of the (here, one dimensional) *Wave Equation*:

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}, \quad t \in [a,b],$$

for some constant $c > 0$. The eigenfunctions of a differential operator related to $\mathcal{L}$ help to solve more general initial value problems of the forms

$$(C):\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{\omega^2(x)}\dfrac{\partial^2 u}{\partial x^2} \\ u(t,a) = u_a(t),\ t>0, \\ u'(t,a) = 0,\ t>0, \end{cases} \qquad (D):\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{\omega^2(x)}\dfrac{\partial^2 u}{\partial x^2} \\ u(t,a) = u(t,b) = 0,\ t>0, \end{cases} \tag{2}$$

where ω is a strictly positive, twice continuously differentiable function s.t. $\int_a^b w(u)du < \infty$.

These equations can be written in a *Normal form* by the change of variables

$$y = \int_0^x \omega(u)du$$

and define $g(y) := \omega^{1/2}(x)f(x)$, $x \in [0, 2\pi]$. We have $\frac{dx}{dy} = \frac{1}{\omega(x)}$. Thus, we have

$$\frac{d^2g}{d^2y} = \Big[\gamma + \frac{\omega''}{2\omega^3} - \frac{3(\omega')^2}{4\omega^4}\Big]g.$$

Some preliminaries

Some preliminaries

A function ψ is said to be an eigenfunction of $\mathcal{L}$ if

$$\mathcal{L}\psi = \lambda\psi, \tag{3}$$

for some number λ (known as the corresponding eigenvalue) and provided that it verifies some specified border conditions.

In particular, it is well known that the eigenvalues are simple, positive and unbounded. That is, the eigenvalues form a sequence

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots \to \infty.$$

Furthermore, the eigenvalues of the operator $\mathcal{L}$ are real and that they verify the asymptotics

$$\begin{cases} \sqrt{\lambda_n} = \frac{n}{2} + O(\frac{1}{n}), & \text{in case (C)} \\ \sqrt{\lambda_n} = \frac{n+1}{2} + O(\frac{1}{n}), & \text{in case (D).} \end{cases}$$

(C) and (D) to be precised soon.

For ease of notation we group the eigenfunctions according to their phase $\frac{n}{2}$.

Some preliminaries

Moreover, the eigenfunctions ψ_n of $\mathcal{L}$, corresponding to the eigenvalues $\lambda_n : n \in \mathbb{N}$, form an orthogonal system, thus generalizing the trigonometric case.

Besides, the eigenfunctions of $\mathcal{L}$ verify the asymptotics (w.l.o.g. we choose the leading constant to be 1, afterwards we will normalize the polynomials so that they have variance one):

- in case (C) : for n large enough, uniformly in $[0, 2\pi]$, we have

$$u_n(x) = \frac{1}{\omega(x)^{1/2}} \cos\left(\frac{n}{2}\int_0^x \omega(u)du\right) + O\Big(\frac{1}{n}\Big),$$

$$u_n'(x) = -\frac{n}{2}\omega(x)^{1/2} \sin\left(\frac{n}{2}\int_0^x \omega(u)du\right) + O(1).$$

- In case (D), for n large enough, uniformly in $[0, 2\pi]$, we have

$$v_n(x) = \frac{1}{\omega(x)^{1/2}} \sin\left(\frac{n}{2}\int_0^x \omega(u)du\right) + O\Big(\frac{1}{n}\Big),$$

$$v_n'(x) = \frac{n}{2}\omega(x)^{1/2} \cos\left(\frac{n}{2}\int_0^x \omega(u)du\right) + O(1).$$

Some preliminaries

Moreover, the eigenfunctions ψ_n of $\mathcal{L}$, corresponding to the eigenvalues $\lambda_n : n \in \mathbb{N}$, form an orthogonal system, thus generalizing the trigonometric case.

Besides, the eigenfunctions of $\mathcal{L}$ verify the asymptotics (w.l.o.g. we choose the leading constant to be 1, afterwards we will normalize the polynomials so that they have variance one):

- in case (C) : for n large enough, uniformly in $[0, 2\pi]$, we have

$$u_n(x) = \frac{1}{\omega(x)^{1/2}} \cos\left(\frac{n}{2}\int_0^x \omega(u)du\right) + O\Big(\frac{1}{n}\Big),$$
$$u_n'(x) = -\frac{n}{2}\omega(x)^{1/2} \sin\left(\frac{n}{2}\int_0^x \omega(u)du\right) + O(1).$$

- In case (D), for n large enough, uniformly in $[0, 2\pi]$, we have

$$v_n(x) = \frac{1}{\omega(x)^{1/2}} \sin\left(\frac{n}{2}\int_0^x \omega(u)du\right) + O\Big(\frac{1}{n}\Big),$$
$$v_n'(x) = \frac{n}{2}\omega(x)^{1/2} \cos\left(\frac{n}{2}\int_0^x \omega(u)du\right) + O(1).$$

Problem setting and main results

Problem setting and main results

Let ω be a strictly positive, twice continuously differentiable weight function s.t.

$$\int_0^{2\pi} \omega(u)du = 2\pi.$$

Consider a sequence of independent standard Gaussian r.v.s $\{a_k, b_k\}_{k=1}^{\infty}$ and define

$$F_n(x) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k u_k(x) + b_k v_k(x), \ \ x \in [0, 2\pi], \tag{4}$$

where $u_k, \ v_k : k \geq 1$ stand for sequences of orthonormal eigenfunctions (with eigenvalue λ_k) of the Sturm-Liouville operator $\mathcal{L}$, (1)-(3), associated to $q = \frac{\omega''}{2\omega^3} - \frac{3}{4}\frac{(\omega')^2}{\omega^4}$, corresponding respectively to two sets of 'basic' initial conditions

$$(C): \ g(0) \neq 0, g'(0) = 0, \qquad (D): \ g(0) = g(2\pi) = 0.$$

.

Let $\mathcal{N}_n$ be the number of roots of F_n on the interval $[0, 2\pi]$, i.e:

$$\mathcal{N}_n := \#\{x \in [0, 2\pi] : F_n(x) = 0\}.$$

The next theorem is the main result of this talk.

Teorema

With the above notation, as $n \to \infty$, there exists $0 < V < \infty$ s.t.

$$\lim_{n\to\infty} \frac{\mathbf{Var}\ (\mathcal{N}_n)}{n} = V,$$

and after normalization, the distribution of $\mathcal{N}_n$ converges towards the standard Gaussian law.

The main idea of the proof of the Theorem is to take profit of the available central limit theorem for the number of zeros of stationary trigonometric polynomials T_n by assesing the L^1 contiguity between both numbers of zeros. As a by-product of our proof we obtain the following *robustness* result for *perturbed* random trigonometric polynomials.

Corolario

Let $\varepsilon_k, \eta_k : [0, 2\pi] \to \mathbb{R}$, $k \geq 1$ be of class C^2 with $|\varepsilon_k(\cdot)|, |\eta_k(\cdot)| \leq \frac{\text{cst}}{k}$ and $|\varepsilon'_k(\cdot)|, |\eta'_k(\cdot)| \leq \text{cst}$, then, the conclusions of Theorem **??** hold true for the number of roots of the perturbed random trigonometric polynomial

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k(\cos(kx) + \varepsilon_k(x)) + b_k(\sin(kx) + \eta_k(x)),$$

for i.i.d standard Gaussian $a_k, b_k : k \geq 1$.

Sketch of the proof

Sketch of the proof

The idea of the proof is to compare the number of roots of F_n with those of Qualls' random trigonometric polynomials T_n.

This is done in two steps

1. To consider the trigonometric part.
2. To consider the *error term*.

Sketch of the proof - step 1

For a process Z_n defined on the interval I, denote its number of zeros and its standardized number of zeros respectively by

$$\mathcal{N}(Z_n, I) := \#\{x \in I : Z_n(x) = 0\} \quad \text{and} \quad \tilde{\mathcal{N}}(Z_n, I) := \frac{\mathcal{N}(Z_n, I) - \mathbf{E}\,\mathcal{N}(Z_n, I)}{\sqrt{n}}.$$

For $x \in [0, 2\pi]$, denote $\Omega(x) := \int_0^x \omega(u)du$, and

$$c_k := \frac{1}{\omega(x)^{1/2}} \cos\left(\frac{k}{2}\Omega(x)\right), \qquad s_k := \frac{1}{\omega(x)^{1/2}} \sin\left(\frac{k}{2}\Omega(x)\right). \tag{5}$$

Let also

$$X_n^o(x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k c_k(x) + b_k s_k(x), \quad x \in [0, 2\pi].$$

Sketch of the proof - step 1

Since $\Omega : [0, 2\pi] \to [0, 2\pi]$ is bijective, setting $y = \Omega(x)$ and introducing the process

$$Y_n^o(y) = X_n^o(\Omega^{-1}(y)) = \sqrt{\frac{2}{n\omega(\Omega^{-1}(y))}} \sum_{k=1}^{n} a_k \cos\left(\frac{k}{2}y\right) + b_k \sin\left(\frac{k}{2}y\right),$$

we have

$$\mathcal{N}(X_n^o, [0, 2\pi]) = \mathcal{N}(Y_n^o, [0, 2\pi]).$$

Thus, we are led to study the number of roots of a *stationary trigonometric polynomial* T_n restricted to the interval $[0, \pi]$.

Sketch of the proof - step 2

The next step is to approximate $\mathcal{N}_n = \mathcal{N}(F_n, [0, 2\pi])$ by $\mathcal{N}(X_n^o, [0, 2\pi])$.

It is convenient to standardize the processes X_n^o and F_n. Define for $x \in [0, 2\pi]$

$$X_n(x) := \omega(x)^{1/2} X_n^o(x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \cos\left(\frac{k}{2}\Omega(x)\right) + b_k \sin\left(\frac{k}{2}\Omega(x)\right),$$

and

$$f_n(x) := \sqrt{\omega(x)} F_n(x).$$

Observe that, since the factor $\sqrt{\omega(x)}$ in the definition of f_n plays no role in the study of the zeros, we have $\mathcal{N}_n = \mathcal{N}(f_n, [0, 2\pi])$.

Sketch of the proof - step 2

The next proposition provides the final approximation we need. Once iestablished, the central limit theorem for $\mathcal{N}(f_n, [0, 2\pi])$ follows from that for $\mathcal{N}(X_n, [0, 2\pi])$.

Proposición

For X_n, f_n defined as above, we have

$$\frac{\mathcal{N}(f_n, [0, 2\pi]) - \mathcal{N}(X_n, [0, 2\pi])}{\sqrt{n}} \xrightarrow[n]{L^1} 0.$$

Observe that this fact implies the CLT.

Sketch of the proof - step 2

For brevity, we set $\mathcal{N}_{f_n}$ and $\mathcal{N}_{X_n}$ for $\mathcal{N}(f_n, [0, 2\pi])$ and $\mathcal{N}(X_n, [0, 2\pi])$ respectively.

We use the Kac formula to estimate the L^1 distance. We have

$$\begin{aligned}
\mathbf{E}\, |\mathcal{N}_{f_n} - \mathcal{N}_{X_n}| &= \mathbf{E}\, \left| \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_0^{2\pi} \left[|X_n'| \mathbb{I}_{|X_n|<\delta} - |f_n'| \mathbb{I}_{|f_n|<\delta} \right] dx \right| \\
&\leq \lim_{\delta \downarrow 0} \frac{1}{2\delta} \mathbf{E} \int_0^{2\pi} \left[|X_n' - f_n'| \mathbb{I}_{|f_n|<\delta} + |X_n'| \left| \mathbb{I}_{|X_n|<\delta} - \mathbb{I}_{|f_n|<\delta} \right| \right] dx \\
&=: A(n) + B(n).
\end{aligned}$$

Sketch of the proof - step 2

We only say a word about $B(n)$. We have

$$
\begin{aligned}
B(n) &= \lim_{\delta\downarrow 0} \frac{1}{2\delta} \int_0^{2\pi} \mathbf{E} \left[|\tilde{X}_n' + \alpha_n f_n| \left| \mathbb{I}_{|X_n|<\delta} - \mathbb{I}_{|f_n|<\delta} \right| \right] dx \\
&\leq \lim_{\delta\downarrow 0} \frac{1}{2\delta} \int_0^{2\pi} \left[\mathbf{E} \, |\tilde{X}_n'| + \alpha \mathbf{E} \, |f_n| \right] \\
&\qquad\qquad \cdot \mathbb{P} \Big\{ \{ |X_n| < \delta, |f_n| > \delta \} \cup \{ |X_n| > \delta, |f_n| < \delta \} \Big\} dx.
\end{aligned}
$$

Here $\tilde{X}_n'$ is the regressed version of X_n'. We have $\mathbf{E} \, |\tilde{X}_n'| \sim_n \mathbf{E} \, |X_n'| = \mathrm{cst} \; n$ and $\mathbf{E} \, |f_n| \sim_n \mathrm{cst}$.

Sketch of the proof - step 2

Denote

$$\varepsilon_n(x) := f_n(x) - X_n(x).$$

We begin with the help of a control in $\|\varepsilon_n\|_\infty := \sup_{x\in[0,2\pi]} |\varepsilon_n(x)|$.

$$\begin{aligned}
&\frac{1}{2\delta}\mathbb{P}\Big\{|X_n| < \delta, |f_n| > \delta, \|\varepsilon_n\|_\infty < \frac{\text{cst } \log n}{\sqrt{n}}\Big\} \\
&\quad \leq \frac{1}{2\delta}\mathbb{P}\Big\{|X_n| < \delta < |f_n| < \delta + \frac{\text{cst } \log n}{\sqrt{n}}\Big\} \\
&\quad = \frac{1}{2\delta}\int_{-\delta}^{\delta} du \int_{\delta}^{\delta+\frac{\text{cst } \log n}{\sqrt{n}}} p_{X_n,f_n}(u,v)dv \underset{\delta\downarrow 0}{\to} \int_0^{\frac{\text{cst } \log n}{\sqrt{n}}} p_{X_n,f_n}(0,v)dv \\
&\quad = \frac{\text{cst } \log n}{\sqrt{n}} p_{X_n,f_n}\Big(0, \theta\frac{\text{cst } \log n}{\sqrt{n}}\Big) \\
&\quad = \frac{\text{cst } \log n}{\sqrt{n}} \frac{1}{2\pi\sqrt{\Delta}} \exp\Big\{-\frac{1}{2}\frac{\mathbf{Var}\ (f_n)\cdot(\frac{\text{cst } \log n}{\sqrt{n}})^2}{\Delta}\Big\}.
\end{aligned}$$

Sketch of the proof - step 2

Here Δ stands for the determinant of $\mathbf{Var}\ (X_n(x), f_n(x))$. As $\Delta \sim_n \frac{\text{cst}}{n}$, we have

$$\lim_{\delta \downarrow 0} \frac{1}{2\delta} \mathbb{P}\Big\{ |X_n| < \delta, |f_n| > \delta, \|\varepsilon_n\|_\infty < \frac{\text{cst}\ \log n}{\sqrt{n}} \Big\}$$
$$\leq \text{cst}\ \log n \cdot \exp\{-\text{cst}\ (\log n)^2\} = \frac{\text{cst}\ \log n}{n^{\text{cst}\ \log n}}.$$

¡Gracias por su atención!

Referencias

1. Angst, Jürgen; Pautrel, Thibault; Poly, Guillaume. Real zeros of random trigonometric polynomials with dependent coefficients. Trans. Amer. Math. Soc. 375 (2022), no. 10, 7209–7260.
2. Angst, Jürgen; Poly, Guillaume. On the absolute continuity of random nodal volumes. Ann. Probab. 48 (2020), no. 5, 2145–2175.
3. Do, Yen; Nguyen, Hoi H; Nguyen, Oanh; Pritsker, Igor. Central limit theorem for the number of real roots of random orthogonal polynomials. `arxiv.org/abs/2111.09015`.
4. Fulton, Charles T.; Pruess, Steven. A. Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems. J. Math. Anal. Appl. 188 (1994), no. 1, 297–340.
5. Granville, Andrew; Wigman, Igor. The distribution of the zeros of random trigonometric polynomials. Amer. J. Math. 133 (2011), no. 2, 295–357.
6. Titchmarsh, E. C. Eigenfunction expansions associated with second-order differential equations. Part I. Second Edition Clarendon Press, Oxford 1962 vi+203 pp.