Introduction to Propagation of Chaos and Mean-Field Systems

Antoine Diez

Kyoto University Institute for the Advanced Study of Human Biology (ASHBi)

Joint work with Louis-Pierre Chaintron

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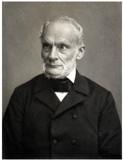
"From the atomistic view to the laws of motion of continua"...

David Hilbert

1. A short historical introduction

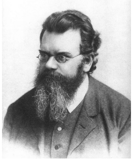
2. Mean-field particle systems and propagation of chaos

The *Stoßzahlansatz* and the early kinetic theory of gas



Clausius

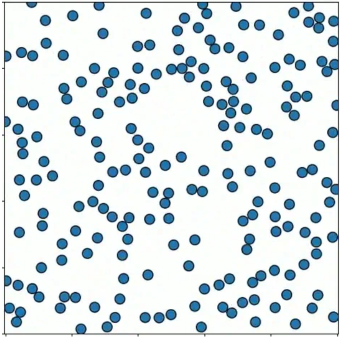
Maxwell



Boltzmann

The hard-sphere gas

A simple mechanical model?



The fundamental assumption

The velocities of two colliding particles are uncorrelated.

- The Maxwellian is the equilibrium distribution
- The entropy increases (H-theorem).
- The time evolution of the distribution of velocities is given by the Boltzmann equation.

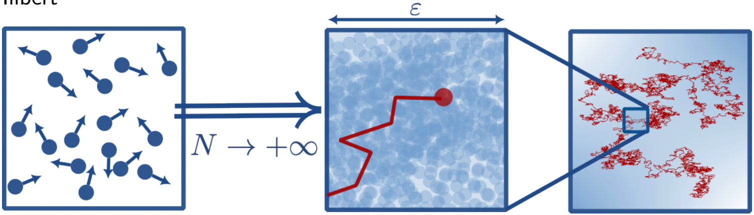
This **cannot** be true for at least two reasons:

1. The collisions create correlations.
2. The Boltzmann equation is irreversible.

David Hilbert

Hilbert 6th problem (1900)

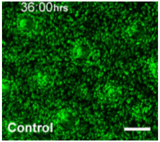
Developing mathematically the limiting processes [. . .] which lead from the atomistic view to the laws of motion of continua.



Microscopic scale.
N identical particles in a space E.
Nd-dimensional dynamical system.

Mesoscopic scale when $N \to +\infty$.
$f_t \in \mathcal{P}(E)$ distribution of a typical particle.
Compute the evolution of statistical quantities

Goal: extend this framework to other types of particle systems. . .



Collisional and mean-field models

Tatyana and Paul Ehrenfest

The Conceptual Foundations of the Statistical Approach in Mechanics (1912)

From the *Stoßzahlansatz* to the *molekular Unordnung*: a statistical point of view.

$\longrightarrow$ The rigorous formulation of Boltzmann's ideas for the hard-sphere gas remains **extremely difficult**: the best available results are only valid in a very **dilute regime** [Grad, 1963] and for **short times** [Lanford, 1975], [Gallagher, Saint-Raymond, Texier, 2014].

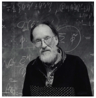
An alternative approach: (stochastic) mean-field models. . .

$\longrightarrow$ Point particles in a dense regime with continuous rescaled interactions by $1/N$.

Mark Kac

Foundations of Kinetic Theory (1956)

Probabilistic interpretation of the Boltzmann equation and the mathematical notion of ***propagation of chaos***.



Henry P. McKean

Propagation of chaos for a class of non-linear parabolic equations (1967)

Extension of Kac ideas to diffusion and other stochastic particle models.

"From the atomistic view to the laws of motion of continua"...

David Hilbert

1. A short historical introduction

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Kac theory: stochastic exchangeable particle systems

Definition: N-particle system

Given a state space E, a N-particle system is a E^N-valued Markov process $\mathcal{X}_t^N = (X_t^1, \dots, X_t^N)$. Its law at time t is denoted by $f_t^N \in \mathcal{P}(E^N)$ and is characterized by the (weak-forward) **Kolmogorov equation**:

$$\forall \varphi_N \in C_b(E^N), \;\; \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\varphi_N(\mathcal{X}_t^N)\big] \equiv \frac{\mathrm{d}}{\mathrm{d}t}\langle f_t^N, \varphi_N\rangle = \langle f_t^N, \mathcal{L}_N \varphi_N\rangle,$$

where $\mathcal{L}_N : C_b(E^N) \to C_b(E^N)$ is the **Markov generator**.

Assumption: Indistinguishability

The process is symmetric: $\forall \pi \in \mathfrak{S}_N, \; (X_t^{\pi(1)}, \dots, X_t^{\pi(N)}) \sim (X_t^1, \dots, X_t^N)$. The N-particle system can thus be represented by its **empirical measure**

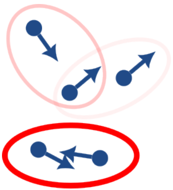
$$\mu_{\mathcal{X}_t^N} := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} \in \mathcal{P}(E).$$

This is a **random measure** whose law is denoted by $F_t^N \in \mathcal{P}(\mathcal{P}(E))$.

About exchangeability, independence, random measures: [Dawson, *St Flour 1991*]

Kac collision processs

Kac model



"Rare stochastic collisions"

A mean-field stochastic collision process

Consider a stochastic Poisson process on each pair of particles with "collision" rate $\lambda(Z_t^i, Z_t^j)/N$.

E.g. : Hard-sphere gas $\lambda(x, y) = \delta_{|x-y|=2R}$.

Collision event at time t, post-collisional states:

$$Z_{t+}^i, Z_{t+}^j \sim \Gamma^{(2)}(Z_t^i, Z_t^j, \mathrm{d}z, \mathrm{d}z^*)$$

Note: $\Gamma^{(2)}(z_1, z_2, \mathrm{d}z_1', \mathrm{d}z_2') = \Gamma^{(2)}(z_2, z_2, \mathrm{d}z_2', \mathrm{d}z_1')$.

Two-particle Markov generator: for $\varphi_2 \in C_b(E^2)$,

$$L^{(2)}\varphi_2(z_1, z_2) = \lambda(z_1, z_2) \int_{E^2} \Big\{\varphi_2(z_1', z_2') - \varphi_2(z_1, z_2)\Big\} \Gamma^{(2)}(z_1, z_2, \mathrm{d}z_1', \mathrm{d}z_2').$$

N-particle Markov generator: for $\varphi_N \in C_b(E^N)$,

$$\mathcal{L}_N \varphi_N = \frac{1}{N} \sum_{i<j} L^{(2)} \diamond_{ij} \varphi_N.$$

$$L^{(2)} \diamond_{ij} \varphi_N(z_1, \ldots, z_N) := L^{(2)}[(u_i, u_j) \mapsto \varphi_N(z_1, \ldots, u_i, \ldots, u_j, \ldots z_N)](z_i, z_j).$$

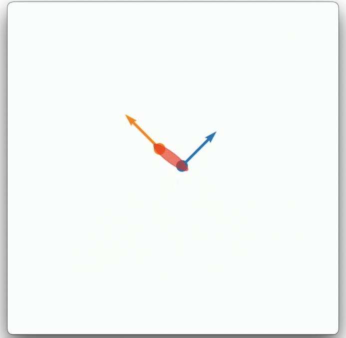
Kac model: example

The random collision time T between two particles depends on the distance:

$$\mathbb{P}(T \geq t) = \mathrm{e}^{-\int_0^t \lambda(|X_s^1 - X_s^2|)\mathrm{d}s}$$

with $r \mapsto \lambda(r)$ non-increasing and new velocities are sampled randomly.

Collision (likely)

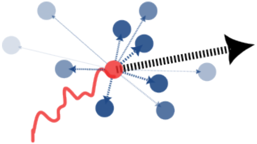


No collision (rare but possible)

Other applications: social sciences, games, opinion dynamics, economics. . .

McKean-Vlasov diffusion model

McKean-Vlasov model



"Small deterministic binary forces plus individual noise"

A mean-field diffusion process

Each particle i feels a small force of size $1/N$ from each of the other particles:

$$\mathrm{d}X_t^i = F \star \mu_{\mathcal{X}_t^N}(X_t^i)\mathrm{d}t + \sigma \mathrm{d}B_t^i.$$

where $F \star \mu(x) := \int_E F(y-x)\mu(\mathrm{d}y)$.

One-particle Markov generator: for $\varphi \in C_b(E)$, $\mu \in \mathcal{P}(E)$,

$$L_\mu \varphi(x) := F \star \mu(x) \cdot \nabla\varphi + \frac{1}{2}\sigma^2 \Delta\varphi$$

N-particle Markov generator: for $\varphi_N \in C_b(E^N)$, $\boldsymbol{x}^N = (x^1, \dots, x^N)$,

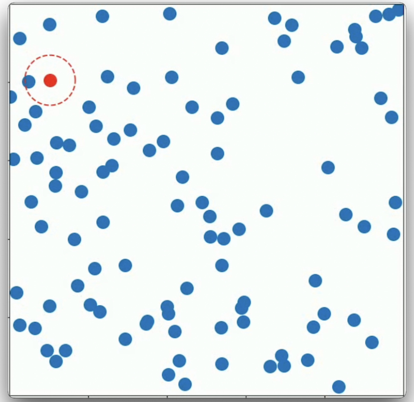
$$\mathcal{L}_N \varphi_N(\boldsymbol{x}^N) = \sum_{i=1}^N L_{\mu_{\boldsymbol{x}^N}} \diamond_i \varphi_N(\boldsymbol{x}^N),$$

with $L_\mu \diamond_i \varphi_N(x^1, \dots, x^N) := L_\mu[u_i \mapsto \varphi_N(x^1, \dots, u_i, \dots, x^N)](x^i)$.

McKean-Vlasov model: example

Self-propulsion and short-range repulsion:

$$\frac{\mathrm{d}X_t^i}{\mathrm{d}t} = V_t^i, \quad \frac{\mathrm{d}V_t^i}{\mathrm{d}t} = (1 - |V_t^i|^2)V_t^i - \frac{1}{N}\sum_{j=1}^{N} \nabla_{x^i}\, \mathrm{e}^{-|X_t^j - X_t^i|/R}.$$



Propagation of chaos

Definition: Kac chaos at a fixed time t

The N-particle distribution f_t^N is **f_t-chaotic** for a given distribution $f_t \in \mathcal{P}(E)$ when for any $s \in \mathbb{N}$, the s-th marginal $f_t^{s,N} \in \mathcal{P}(E^s)$ of $f_t^N \in \mathcal{P}(E^N)$ satisfies

$$f_t^{s,N} \underset{N\to+\infty}{\longrightarrow} f_t^{\otimes s} \quad \text{weakly in } \mathcal{P}(E^s).$$

$\longrightarrow$ *" When N is large, any group of s particles is close to be independent."*

Definition: Propagation of chaos

It means: f_0^N is f_0-chaotic **implies** f_t^N is f_t-chaotic for $t \geq 0$.

From now on, $\boxed{f_0^N = f_0^{\otimes N}}$ (the particles are initially i.i.d.).

Lemma

The two following assertions are **equivalent** to Kac chaos.

(i) Kac chaos for the marginal $s = 2$, i.e. $f_t^{2,N} \to f_t^{\otimes 2}$ weakly in $\mathcal{P}(E)$.

(ii) The empirical measure process converges in law towards the deterministic measure f_t, i.e. $\forall \Phi \in C_b(\mathcal{P}(E))$, $\quad \mathbb{E}\big[\Phi\big(\mu_{\mathcal{X}_t^N}\big)\big] \to \Phi(f_t)$.

The two building block theorems of Kac and McKean.

1. Kac theorem: Markov generator and series expansion
2. McKean theorem: empirical measure, stochastic paths, coupling
3. Variations and alternative points of view

Kac theorem [Kac, 1956], [Carlen, Degond, Wennberg, *Ma. Mo. Me. Ap. Sc. 23*, 2013]

Recall: $\mathcal{L}_N\varphi_N = \frac{1}{N}\sum_{i<j} L^{(2)} \diamond_{ij} \varphi_N$ and $L^{(2)}$ is a two-particle jump operator.

Cut-off assumption

The operator $L^{(2)}$ is **bounded** in L^∞. E.g. the collision rate $\lambda \equiv 1$ is **constant**.

Consequence: $\mathcal{L}_N$ is bounded in L^∞ and for $\varphi_s \equiv \varphi_s \otimes 1^{N-s} \in C_b(E^s) \subset C_b(E^N)$

$$\langle f_t^{s,N}, \varphi_s\rangle = \langle f_0^N, \mathrm{e}^{t\mathcal{L}_N}\varphi_s\rangle = \sum_{k=0}^{+\infty} \frac{t^k}{k!}\langle f_0^N, \mathcal{L}_N^k \varphi_s\rangle.$$

$\longrightarrow$ Take the limit $N \to +\infty$ of each term **uniformly in** t.

Main observation for $k = 1$

$$\langle f_0^N, \mathcal{L}_N\varphi_s\rangle = \frac{s}{N}\langle f_0^{s,N}, \mathcal{L}_s\varphi_s\rangle + \frac{N-s}{N}\langle f_0^{s+1,N}, \mathbf{D}\varphi_s\rangle,$$

where the operator $\mathbf{D} : C_b(E^s) \to C_b(E^{s+1})$ is defined by:

$$\mathbf{D}\varphi_s = \sum_{i=1}^{s} L^{(2)} \diamond_{i,s+1} (\varphi_s \otimes 1).$$

Note: hierarchy structure or "recollision tree" [Graham, Méléard, *Ann. Prob. 25*, 1997]

The main lemma

For $k > 1$, the same structure holds and under the initial chaos assumption $f_0^s = f_0^{\otimes s}$,

$$\langle f_0^N, \mathcal{L}_N^k \varphi_s \rangle \underset{N\to+\infty}{\longrightarrow} \langle f_0^{\otimes(s+k)}, \mathbf{D}^k \varphi_s \rangle.$$

Moreover the series converges absolutely uniformly in N on $(0, t_0)$.

Consequently, this **defines** a limit distribution $f_t^{s,\infty} \in \mathcal{P}(E^s)$ by:

$$\langle f_t^{s,N}, \varphi_s \rangle = \sum_{k=0}^{+\infty} \frac{t^k}{k!} \langle f_0^N, \mathcal{L}_N^k \varphi_s \rangle \underset{N\to+\infty}{\longrightarrow} \sum_{k=0}^{+\infty} \frac{t^k}{k!} \langle f_0^{\otimes(s+k)}, \mathbf{D}^k \varphi_s \rangle =: \langle f_t^{s,\infty}, \varphi_s \rangle.$$

It remains to prove that $f_t^{s,\infty} = f_t^{\otimes s}$ where $f_t = f_t^{1,\infty} \dots$

This follows from Leibniz formula and the following observation (due to McKean)

$$\mathbf{D}(\varphi_{s_1} \otimes \varphi_{s_2}) = \mathbf{D}\varphi_{s_1} \otimes \varphi_{s_2} + \varphi_{s_1} \otimes \mathbf{D}\varphi_{s_2}.$$

$$\begin{aligned}\langle f_t^{s_1+s_2,\infty}, \varphi_{s_1} \otimes \varphi_{s_2} \rangle &= \sum_{k=0}^{+\infty} \frac{t^k}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} \langle f_0^{\otimes(s_1+s_2+k)}, \mathbf{D}^\ell \varphi_{s_1} \otimes \mathbf{D}^{k-\ell} \varphi_{s_2} \rangle \\ &= \langle f_t^{s_1,\infty}, \varphi_{s_1} \rangle \langle f_t^{s_2,\infty}, \varphi_{s_2} \rangle.\end{aligned}$$

Kac theorem [Kac, 1956], [Carlen, Degond, Wennberg, *Ma. Mo. Me. Ap. Sc. 23*, 2013]

Computing $\frac{\mathrm{d}}{\mathrm{d}t}\langle f_t, \varphi\rangle = \sum_{k=0}^{+\infty} \frac{t^k}{k!}\langle f_0^{\otimes(s+2)}, \mathbf{D}^k[\mathbf{D}\varphi]\rangle = \langle f_t^{2,\infty}, \mathbf{D}\varphi\rangle$ leads to:

Theorem: The Kac-Boltzmann equation

For any $s \leq N$, $f_t^{s,N} \to f_t^{\otimes s}$ and the limit law f_t satisfies for all $\varphi \in C_b(E)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f_t, \varphi\rangle = \langle f_t^{\otimes 2}, \mathbf{D}\varphi\rangle.$$

Recall,

$$\langle f_t^{\otimes 2}, \mathbf{D}\varphi\rangle = \int_{E^3} \{\varphi(z_1') - \varphi(z_1)\}\Gamma^{(2)}(z_1, z_2, \mathrm{d}z_1', E) f_t(\mathrm{d}z_1) f_t(\mathrm{d}z_2).$$

$\longrightarrow$ The equation is written in **weak form**, the strong form $\partial_t f_t = Q(f_t, f_t)$ can (sometimes) be obtained by computing the dual operator

$$\mathbf{D}^* : \mathcal{P}(E^2) \to \mathcal{P}(E).$$

$\longrightarrow$ In the final equation only the marginal $\Gamma^{(2)}(z_1, z_2, \mathrm{d}z_1', E)$ appears which means that **the details of the interaction mechanism is lost in the limit**: different Kac processes can have the same mean-field limit.

The two building block theorems of Kac and McKean.

1. Kac theorem: Markov generator and series expansion
2. McKean theorem: empirical measure, stochastic paths, coupling
3. Variations and alternative points of view

First step: guess the limit

Recall: $\mathcal{L}_N \varphi_N(\boldsymbol{x}^N) = \sum_{i=1}^N L_{\mu_{\boldsymbol{x}^N}} \diamond_i \varphi_N(\boldsymbol{x}^N)$ where L_μ is a one-particle operator.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f_t^{1,N}, \varphi\rangle = \int_E L_{\mu_{\boldsymbol{x}^N}} \varphi(x^1) f_t^N(\mathrm{d}\boldsymbol{x}^N) = \int_E \left(\frac{1}{N}\sum_{i=1}^N L_{\mu_{\boldsymbol{x}^N}}\varphi(x^i)\right) f_t^N(\mathrm{d}\boldsymbol{x}^N)$$

$$= \int_E \langle \mu_{\boldsymbol{x}^N}, L_{\mu_{\boldsymbol{x}^N}}\varphi\rangle f_t^N(\mathrm{d}\boldsymbol{x}^N) = \int_{\mathcal{P}(E)} \langle \mu, L_\mu \varphi\rangle F_t^N(\mathrm{d}\mu).$$

If $\mu_{\mathcal{X}_t^N} \to f_t$ then $f_t^{1,N} = \mathbb{E}\mu_{\mathcal{X}_t^N} \to f_t$ and f_t satisfies the **nonlinear equation**

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle f_t, \varphi\rangle = \langle f_t, L_{f_t}\varphi\rangle \quad \text{i.e.} \quad \partial_t f_t = L_{f_t}^\star f_t.}$$

Fokker-Planck equation

Drift $b(x,\mu) = F \star \mu(x)$ and diffusion matrix $\sigma(x,\mu)$,

$$L_\mu \varphi(x) = b(x,\mu)\cdot\nabla\varphi + \frac{1}{2}\sum_{i,j=1}^d \sigma_{ij}\sigma_{ij}^{\mathrm{T}}(x,\mu)\,\partial_{x^i}\partial_{x^j}\varphi.$$

$$\partial_t f_t = -\nabla_x\cdot(b(x,f_t)f_t) + \frac{1}{2}\sum_{i,j=1}^d \partial_{x^i}\partial_{x^j}\{\sigma_{ij}\sigma_{ij}^{\mathrm{T}}(x,f_t)f_t\}.$$

Wasserstein distance between marginals and coupling

Definition: Wasserstein distance

Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures with bounded second moment. Then for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$W_2^2(\mu, \nu) := \inf_{X \sim \mu,\ Y \sim \nu} \mathbb{E}|X - Y|^2$$

defines a distance on $\mathcal{P}_2(\mathbb{R}^d)$ which **metrizes** the weak convergence.

In particular let $\overline{X}_t^i \sim f_t$, $i \in \{1, \dots, N\}$ be N i.i.d. random variables, then

$$W_2^2(f_t^{s,N}, f_t^{\otimes s}) \leq \sum_{i=1}^{s} \mathbb{E}|X_t^i - \overline{X}_t^i|^2 = s\, \mathbb{E}\left|X_t^1 - \overline{X}_t^1\right|^2$$

Everything boils down to proving that

$$\mathbb{E}\left|X_t^1 - \overline{X}_t^1\right|^2 \underset{N \to +\infty}{\longrightarrow} 0 \dots$$

... for some $X_t^1, \overline{X}_t^1 \sim f_t^{1,N}, f_t$ that can be **constructed as one wishes**.

$\longrightarrow$ Such random variables are called a **coupling**.

Note: $\mathbb{E} W_2^2(\mu_{\mathcal{X}_t^N}, f_t) \to 0$ also implies $F_t^N \to \delta_{f_t}$.

McKean Theorem in the bounded Lipschitz case

Consider the McKean-Vlasov model with an **arbitrary** drift $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$,

$$\mathrm{d}X_t^i = b\big(X_t^i, \mu_{\mathcal{X}_t^N}\big)\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i.$$

Introduce the **synchronous coupling** with N independent **nonlinear processes**

$$\mathrm{d}\overline{X}_t^i = b\big(\overline{X}_t^i, f_t\big)\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \quad \overline{X}_0^i = X_0^i.$$

where $f_t = \mathrm{Law}(\overline{X}_t^i)$ satisfies the nonlinear Fokker-Planck equation:

$$\partial_t f_t = -\nabla_x \cdot \big(b(x, f_t) f_t\big) + \Delta_x f_t.$$

Theorem (well-posedness)

Let b be **bounded and Lipschitz** for the W_2 distance:

$$|b(x,\mu) - b(y,\nu)| \leq C\Big(|x-y| + W_2(\mu,\nu)\Big).$$

Then for any $T > 0$, the nonlinear Fokker-Planck equation is well-posed in $C([0,T], \mathcal{P}_2(\mathbb{R}^d))$ and the associated SDE has a unique strong solution.

Theorem (McKean)

$$\forall T > 0, \ \lim_{N \to +\infty} \mathbb{E}\Big[\sup_{t \leq T} \big|X_t^i - \overline{X}_t^i\big|^2\Big] = 0.$$

McKean Theorem: proof (1/2) [Sznitman, *St Flour*, 1989]

By construction and the BDG inequality (…or Itō lemma), for $i \in \{1, \dots, N\}$,

$$\mathbb{E}\Big[\sup_{t\leq T} |X_t^i - \overline{X}_t^i|^2\Big] \leq 2T \int_0^T \mathbb{E}\Big|b\big(X_t^i, \mu_{\mathcal{X}_t^N}\big) - b\big(\overline{X}_t^i, f_t\big)\Big|^2 \mathrm{d}t$$

$$\leq 4T \int_0^T \mathbb{E}\Big|b\big(X_t^i, \mu_{\mathcal{X}_t^N}\big) - b\big(\overline{X}_t^i, \mu_{\overline{\mathcal{X}}_t^N}\big)\Big|^2 + \mathbb{E}\Big|b\big(\overline{X}_t^i, \mu_{\overline{\mathcal{X}}_t^N}\big) - b\big(\overline{X}_t^i, f_t\big)\Big|^2 \mathrm{d}t$$

where $\mu_{\overline{\mathcal{X}}_t^N} = \frac{1}{N}\sum_{i=1}^N \delta_{\overline{X}_t^i}$. Then,

- $$\mathbb{E}\Big|b\big(X_t^i, \mu_{\mathcal{X}_t^N}\big) - b\big(\overline{X}_t^i, \mu_{\overline{\mathcal{X}}_t^N}\big)\Big|^2 \leq C\Big(\mathbb{E}\big|X_t^i - \overline{X}_t^i\big|^2 + \mathbb{E}W_2^2\big(\mu_{\mathcal{X}_t^N}, \mu_{\overline{\mathcal{X}}_t^N}\big)\Big)$$
  $$\leq C\Big(\mathbb{E}\big|X_t^i - \overline{X}_t^i\big|^2 + \frac{1}{N}\sum_{j=1}^N \mathbb{E}\big|X_t^j - \overline{X}_t^j\big|^2\Big) \leq 2C\,\mathbb{E}\big|X_t^i - \overline{X}_t^i\big|^2.$$
- $$\mathbb{E}\Big|b\big(\overline{X}_t^i, \mu_{\overline{\mathcal{X}}_t^N}\big) - b\big(\overline{X}_t^i, f_t\big)\Big|^2 \leq C\,\mathbb{E}W_2^2\big(\mu_{\overline{\mathcal{X}}_t^N}, f_t\big).$$

In conclusion,

$$\mathbb{E}\Big[\sup_{t\leq T} |X_t^i - \overline{X}_t^i|^2\Big] \leq C_1 \int_0^T \mathbb{E}W_2^2\big(\mu_{\overline{\mathcal{X}}_t^N}, f_t\big)\,\mathrm{d}t + C_2 \int_0^T \mathbb{E}|X_t^i - \overline{X}_t^i|^2\,\mathrm{d}t.$$

McKean Theorem: proof (2/2) [Sznitman, *St Flour*, 1989]

By Gronwall lemma,

$$\mathbb{E}\Big[\sup_{t\le T}|X^i_t - \overline{X}^i_t|^2\Big] \le C_1\,\mathrm{e}^{C_2 T}\int_0^T \mathbb{E}W_2^2\big(\mu_{\overline{\mathcal{X}}^N_t}, f_t\big)\,\mathrm{d}t.$$

- By the strong **Law of Large Numbers**, for some constant $M_2 > 0$,

$$\mu_{\overline{\mathcal{X}}^N_t} \to f_t \text{ a.s.}, \quad \mathbb{E}W_2^2\big(\mu_{\overline{\mathcal{X}}^N_t}, f_t\big) \le M_2 \text{ and } \mathbb{E}W_2^2\big(\mu_{\overline{\mathcal{X}}^N_t}, f_t\big) \to 0.$$

  [Carmona, *Lectures on BSDEs [...]*, SIAM, 2015]

- If f_0 has sufficiently **high-order moments**,

$$\mathbb{E}W_2^2\big(\mu_{\overline{\mathcal{X}}^N_t}, f_t\big) = \begin{cases} \mathcal{O}(N^{-2/d}) & \textit{if } d > 4 \\ \mathcal{O}(N^{-1/2}) & \textit{if } d < 4 \\ \mathcal{O}(N^{-1/2}\log(1+N)) & \textit{if } d = 4 \end{cases}.$$

  [Fournier, Guillin, *Prob. Th. Rel. Fi. 162*, 2015]

- If $b(x,\mu) = F\star\mu(x)$ (or a function of $F\star\mu(x)$) with F bounded Lipschitz,

$$\mathbb{E}\Big|b\big(\overline{X}^i_t, \mu_{\overline{\mathcal{X}}^N_t}\big) - b\big(\overline{X}^i_t, f_t\big)\Big|^2 \le \frac{C}{N}.$$

  [Sznitman, *St Flour*, 1989], [McKean, 1967]

The two building block theorems of Kac and McKean.

1. Kac theorem: Markov generator and series expansion
2. McKean theorem: empirical measure, stochastic paths, coupling
3. Variations and alternative points of view

Variation 1: McKean by McKean (1967)

Synchronous coupling between a N-particle system and a $M > N$ particle system:

$$\mathrm{d}X_t^{i,N} = F \star \mu_{\mathcal{X}_t^N}(X_t^{i,N})\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \quad X_0^{i,N} = X_0^i,$$
$$\mathrm{d}X_t^{i,M} = F \star \mu_{\mathcal{X}_t^M}(X_t^{i,M})\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \quad X_0^{i,M} = X_0^i.$$

1. By the same (slightly simpler) computations: $\mathbb{E}\sup_{t\leq T}|X_t^{i,N} - X_t^{i,M}|^2 \to 0$ when $N, M \to +\infty$.
2. For any i, the process $(X_t^{i,N})_t$ is **Cauchy** in $L^2(\Omega, C([0,T],\mathbb{R}^d))$ and thus there are limit points $(\overline{X}_t^i)_t$ which are identically distributed.
3. By construction
$$\overline{X}_t^i \in \sigma(X_0^1, (B_t^1)_t, X_0^2, (B_t^2)_t, \ldots).$$
However, by **exchangeability** and by **Hewitt-Savage 0-1 law**,
$$\overline{X}_t^i \in \sigma\big(X_0^i, (B_t^i)_t\big),$$
and these processes are thus **independent**.
4. Check that the process $(\overline{X}_t^i)_t$ solves the nonlinear SDE.

Extension: Mean-field jump processes

The generator of mean-field jump processes

$$L_\mu\varphi(x) = a\cdot\nabla\varphi(x) + \lambda(x,\mu)\int_E\{\varphi(y)-\varphi(x)\}P_\mu(x,\mathrm{d}y),$$

where

- $a : E \to E$ deterministic flow $\dot{X}_t = a(X_t)$,
- $\lambda(x,\mu)$ (non-homogeneous) jump frequency,
- $P_\mu(x,\mathrm{d}y)$ law of the post-jump state.

Example (Run-and-tumble motion). $E=\mathbb{R}^d\times\mathbb{R}^d$ with $Z_t^i=(X_t^i,V_t^i)$ and

- $a(x,v)=(v,0)$ (free transport),
- $\lambda(x,\mu)\equiv 1$ constant,
- $P_\mu((x,v),\mathrm{d}x',\mathrm{d}v') = \delta_x(\mathrm{d}x')\otimes\mathscr{M}_{\mu,x}(v')\mathrm{d}v'$ with the Maxwellian,

$$\mathscr{M}_{\mu,x}(v) = \frac{1}{(2\pi T)^{d/2}}\exp\left(\frac{|v-u|^2}{2T}\right),$$

where (ρ,u,T) are defined by $(\rho,\rho u,\rho|u|^2+\rho T) := \int_{\mathbb{R}^d}(1,v,|v|^2)K\star\mu(x,\mathrm{d}v)$.

$\longrightarrow$ **Mean-field limit:** $\partial_t f_t + v\cdot\nabla_x f_t = \rho_{f_t}(x)\mathscr{M}_{f_t,x}(v) - f_t$. (BGK equation)

[Buttà, Hauray, Pulvirenti, *ARMA 240*, 2021], [D., *EJP 25*, 2020]...

SDE representation of mean-field jump and Kac processes

Assume that $P_\mu(x, \mathrm{d}y)$ is **parametrized** by **fixed** parameter probability space $(\Theta, \nu(\mathrm{d}\theta))$ and a given function $\psi : E \times \mathcal{P}(E) \times \Theta \to E$ such that

$$\int_E \varphi(y) P_\mu(x, \mathrm{d}y) = \int_\Theta \varphi\big(\psi(x, \mu, \theta)\big)\nu(\mathrm{d}\theta).$$

$$X_t^i = X_0^i + \int_0^t \int_0^{+\infty} \int_\Theta \left\{\psi\Big(X_{s-}^i, \mu_{\mathcal{X}_{s-}^N}, \theta\Big) - X_{s-}^i\right\} \mathbf{1}_{\left(0, \lambda\left(X_{s-}^i, \mu_{\mathcal{X}_{s-}^N}\right)\right]}(u) \mathcal{N}^i(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\theta)$$

where $\mathcal{N}^i(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\theta)$ are N independent Poisson random measures with intensity $\mathrm{d}s \otimes \mathrm{d}u \otimes \nu(\mathrm{d}\theta)$ on $[0, +\infty) \times [0, +\infty) \times \Theta$.

L^1 framework: [Graham, *Ann. Inst. H. Poincaré 28*, 1992], [Graham, *Sto. Pr. App. 40*, 1992], [Andreis, Dai Pra, Fischer, *Sto. Ana. Appl. 36*, 2018]

Boltzmann-Kac equation. . . (with constant collision rate)
New state function ψ, collision partner α, collision type σ, and non-independent $\mathcal{N}^i$

$$Z_t^i = Z_0^i + \int_0^t \int_\Theta \int_{\{0,1\}} \int_{\{1,\dots,N\}} \left\{\psi_{\sigma}(Z_{s-}^i, Z_{s-}^{\alpha}, \theta) - Z_{s-}^i\right\} \mathcal{N}^i(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}\sigma, \mathrm{d}\alpha).$$

[Tanaka, *Z. Wahr. verw. Geb. 46*, 1978], [Murata, *Hiroshima Math. J. 7*, 1977], [Cortez, Fontbona, *Ann. App. Pro. 26*, 2016], [Cortez, Fontbona, *Comm. Math. Phys. 357*, 2018], [Fournier, Mischler, *Ann. Pro. 44*, 2016]

Pathwise point of view on $I = (0, T)$

- **Pointwise** propagation of chaos holds towards a flow of measures $(f_t)_t \in C(I, \mathcal{P}(E))$ when the law $f_t^N \in \mathcal{P}(E^N)$ of $\mathcal{X}_t^N$ is f_t-chaotic for every time $t \in I$.
- **Pathwise** propagation of chaos holds towards a distribution $f_I \in \mathcal{P}(D(I, E))$ on the space $D(I, E)$ of càdlàg functions when the law $f_I^N \in \mathcal{P}(D(I, E)^N)$ of the process $\mathcal{X}_I^N$ seen as a random element in $D(I, E)^N$ is f_I-chaotic.

Example.

(Pointwise) $$W_2^2(f_t^{s,N}, f_t^{\otimes s}) \leq s\mathbb{E}\left|X_t^1 - \overline{X}_t^1\right|^2 \leq s \sup_{t\in I} \mathbb{E}|X_t^1 - \overline{X}_t^1|^2.$$

(Pathwise) $$\mathcal{W}_2^2(f_I^{s,N}, f_I^{\otimes s}) \leq s\,\mathbb{E}\left\|X_I^1 - \overline{X}_I^1\right\|_{C(I,E)}^2 = s\,\mathbb{E}\left[\sup_{t\in I}\left|X_t^1 - \overline{X}_t^1\right|^2\right].$$

There are **two** pathwise empirical measure processes:

- The measure-valued process $\left(\mu_{\mathcal{X}_t^N}\right)_t$ with law $F_I^{\mu,N} \in \mathcal{P}(D(I, \mathcal{P}(E)))$.
- The empirical measure of the processes $\mu_{\mathcal{X}_I^N}$ with law $F_I^N \in \mathcal{P}(\mathcal{P}(D(I, E)))$.

$[F_I^N \to \delta_{f_I}]$	$\Longrightarrow$	$[F_I^{\mu,N} \to \delta_{(f_t)_t}]$	$\Longrightarrow$	$[F_t^N \to \delta_{f_t}]$
Pathwise p.o.c.		**Functional L.L.N.**		**Pointwise p.o.c.**
$\mathcal{P}(\mathcal{P}(D(I, E)))$		$\mathcal{P}(D(I, \mathcal{P}(E)))$		$\mathcal{P}(\mathcal{P}(E))$

Pathwise point of view: Martingale problems

$\longrightarrow$ The N-particle process is defined as the solution of a **martingale problem**.

Pathwise particle martingale problem

$$\forall \varphi_N \in \mathrm{Dom}(\mathcal{L}_{\mathcal{N}}),\ M_t^{\varphi_N} := \varphi_N\big(\mathbf{X}_t^N\big) - \varphi_N\big(\mathbf{X}_0^N\big) - \int_0^t \mathcal{L}_N \varphi_N\big(\mathbf{X}_s^N\big)\mathrm{d}s,$$

is a f_I^N-martingale, where $\mathbf{X}_t^N(\omega) = \omega(t)$ is the canonical process in $D(I, E^N)$.

$\longrightarrow$ Similarly for the limit nonlinear processes...

Pathwise nonlinear Boltzmann-Kac martingale problem

$$\forall \varphi_N \in C_b(E),\ M_t^{\varphi} := \varphi(\mathsf{X}_t) - \varphi(\mathsf{X}_0) - \int_0^t \langle f_s, \mathbf{D}\varphi(\mathsf{X}_s, \cdot)\rangle\,\mathrm{d}s,$$

is a f_I-martingale, where $\mathsf{X}_t(\omega) = \omega(t)$ and $f_s = (\mathsf{X}_s)_{\#} f_I \in \mathcal{P}(E)$.

Pathwise nonlinear McKean-Vlasov martingale problem

$$\forall \varphi_N \in C_b(E),\ M_t^{\varphi} := \varphi(\mathsf{X}_t) - \varphi(\mathsf{X}_0) - \int_0^t L_{f_s}\varphi(\mathsf{X}_s)\mathrm{d}s,$$

is a f_I-martingale, where $\mathsf{X}_t(\omega) = \omega(t)$ and $f_s = (\mathsf{X}_s)_{\#} f_I \in \mathcal{P}(E)$.

Pathwise point of view: Martingale methods

General outline of the proof

1. Show that $(F_I^N)_N$ is tight using classical **tightness criteria:** Aldous, Rebolledo, Joffe-Métivier...
   By Prokhorov theorem, there exists a limit point $\pi \in \mathcal{P}(\mathcal{P}(D(I, E)))$.
2. Identify the π-distributed limit points as solutions of the limit martingale problem (this provides an **existence** result).
3. Prove the **uniqueness** of the limit martingale problem. This implies that π is a Dirac mass at this point.

A very general methodology!

For the Boltzmann-equation...
[Tanaka, *Proc. IFIP-WG 7/1, Bangalore 1982*, 1983], [Sznitman, *Zeit. Wah. Ver. Geb. 66*, 1984], [Wagner, *Sto. An. App. 14*, 1996]...

For McKean-Vlasov systems and more...
[Sznitman, *J. Fun. An. 56*, 1984], [Oelschläger, *An. Prob. 12*, 1984], [Gärtner, *Math. Nachr. 137*, 1988], [Graham, Méléard, *Ann. Probab. 25*, 1997]...

Some drawbacks: no convergence rate, typically much more technical.

Two important questions and some applications. . .

1. Long-time behaviour and uniform-in-time propagation of chaos

2. Low regularity, singular and abstract interactions

(One) motivation. Understand the long-time behaviour of **mesoscopic nonlinear** systems via their particle representation.

Example: trend to equilibrium for the **granular media equation:**

$$\partial_t f_t = \nabla \cdot \big(f_t \nabla (V + W \star f_t)\big) + \Delta f_t.$$

- V confinement potential (e.g. $V(x) = |x|^2/2$).
- W (symmetric) interaction potential.

[Carrillo, McCann, Villani, *Rev. Ma. Iberoa. 19*, 2003], [Bolley, Gentil, Guillin, *ARMA 208*, 2013]

Mean-field particle representation:

$$\mathrm{d}X_t^i = -\nabla V(X_t^i)\mathrm{d}t - \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_t^j - X_t^i)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t^i.$$

High-dimensional Langevin dynamics with invariant measure:

$$\pi_\infty^N(\mathrm{d}\boldsymbol{x}^N) \propto \exp\left(-\sum_{i=1}^{N} V(x^i) - \frac{1}{2N}\sum_{i,j=1}^{N} W(x^i - x^j)\right)\mathrm{d}x^1 \dots \mathrm{d}x^N.$$

Long-time behaviour: Malrieu's theorem

Earlier work in 1D: [Benachour, Roynette, Vallois, *Sto. Pro. App. 75*, 1998]

Theorem (Malrieu, *Sto. Pro. App. 95*, 2001)

If V is β-uniformly convex, W is symmetric, convex, ∇W is locally Lipschitz with polynomial growth then the synchronous coupling is uniform in time:

$$\sup_{t\geq 0}\mathbb{E}|X_t^i - \overline{X}_t^i|^2 \leq \frac{C}{N}.$$

It implies the exponential convergence of f_t towards a unique invariant measure.

Key idea: With Itō's formula,

$$\frac{\mathrm{d}}{\mathrm{d}t}|X_t^i - \overline{X}_t^i|^2 \leq -2(X_t^i - \overline{X}_t^i)\cdot(\nabla V(X_t^i) - \nabla V(\overline{X}_t^i)) + \ldots \leq -2\beta|X_t^i - \overline{X}_t^i|^2 + \ldots$$

Extensions, and related works...

Non uniformly convex V: [Cattiaux, Guillin, Malrieu, *Pr. Th. Rel. Fi. 140*, 2008]...

Kinetic (2nd order) systems: [Bolley, Guillin, Malrieu, *ESAIM Ma. Mo. Nu. An. 44*, 2010], [Monmarché, *Sto. Pr. App., 127*, 2017]...

New coupling methods: [Durmus, Eberle, Guillin, Zimmer, *Proc. Amer. Math. Soc. 148*, 2020], [Guillin, Le Bris, Monmarché, *EJP 27*, 2022]...

Long-time behaviour: Phase transitions

Unlike the particle system, the mean-field limit can have **several** invariant measures.

The Kuramoto model: synchronization of oscillators $\theta_t^i \in \mathbb{S}^1$ with strength $\gamma > 0$

$$\mathrm{d}\theta_t^i = \frac{\gamma}{N}\sum_{j=1}^N \sin(\theta_t^j - \theta_t^i)\mathrm{d}t + \mathrm{d}B_t^i, \quad \partial_t f_t(\theta) = -\gamma\nabla_\theta\cdot(f_t(\sin\star f_t)) + \frac{1}{2}\Delta f_t$$

Stable invariant measure of the mean-field equation: for $\theta_0 \in \mathbb{R}$,

$$M_{\kappa,\theta_0}(\theta) \propto \exp(-\kappa\cos(\theta-\theta_0)),\ \kappa = 2\gamma\frac{I_1(\kappa)}{I_0(\kappa)}.$$

Phase transition:

- If $\gamma < 1$, $\kappa = 0$ is the unique solution.
- If $\gamma > 1$, there is another solution $\kappa^* > 0$ and M_{κ^*,θ_0} is asymptotically stable.

Long-time behaviour: there exists a Brownian noise $(W_t)_t$

$$\frac{1}{N}\sum_{i=1}^N \delta_{\theta^i_{Nt}} \approx M_{\kappa^*,\,\theta_0+W_t} \neq M_{\kappa^*,\theta_0}.$$

[Bertini, Giacomin, Poquet, *Prob. Th. Rel. Fi. 160*, 2014]

$\longrightarrow$ The propagation of chaos breaks down at time proportional to N.

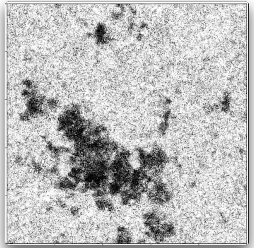
Long-time behaviour: some research directions

A (not so) recent trend: explore the links between phase transitions, uniform in time propagation of chaos and log-Sobolev inequalities...
[Malrieu, *Sto. Pro. App. 95*, 2001], [Delgadino, Gvalani, Pavliotis, *Ar. Ra. Me. An. 241*, 2021], [Delgadino, Gvalani, Pavliotis, Smith, *Comm. Math. Phys.*, 2023], [Guillin, Monmarché, *J. Stat. Phys. 185*, 2021]...

A long-standing problem: Trend to equilibrium for Boltzmann models [Kac, 1956], [Grünbaum, ARMA 42, 1971], [Mischler, Mouhot, *Inv. Math. 193*, 2013]...

An open problem: phase transitions in the Vicsek model (and other kinetic models)

$$\partial_t f_t(x,v) + v \cdot \nabla_x f_t = \text{[\textit{some mean-field operator acting on } v \text{ \textit{with phase transition}}]}.$$



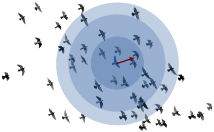
Local alignment + noise

Two important questions and some applications. . .

1. Long-time behaviour and uniform-in-time propagation of chaos
2. Low regularity, singular and abstract interactions

Low-regularity and singular interactions

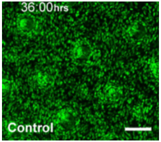
- **Interaction kernel in collective dynamics models**



Flocking in the Cucker-Smale model.

$$\mathrm{d}X_t^i = V_t^i \mathrm{d}t, \;\; \mathrm{d}V_t^i = \frac{1}{N}\sum_{j=1}^{N} \frac{V_t^j - V_t^i}{(1+|X_t^j - X_t^i|^2)^\gamma}\mathrm{d}t + \mathrm{d}B_t^i$$

- **(Overdamped) Keller-Segel and Coulomb-type interactions**



[Glover et al., 2017]

$$\partial_t \rho = -\nabla\cdot(\rho\nabla c) + \frac{1}{2}\Delta\rho, \quad -\Delta c = \rho.$$

$$\mathrm{d}X_t^i = \frac{1}{N}\sum_{j=1}^{N} K(X_t^j - X_t^i)\mathrm{d}t + \mathrm{d}B_t^i, \quad K(r) = \xi\frac{x}{|x|^d}.$$

- **Unbounded jump rates in Boltzmann-Kac and mean-field jumps models**

Spiking neurons rate $\lambda(r) = (r/r_0)^\alpha$ [Fournier, Löcherbach, *Ann. IHP Pr. St. 52*, 2015]

$$X_t^i = X_0^i - \lambda\int_0^t \Big(X_s^i - \frac{1}{N}\sum_{j=1}^{N} X_s^j\Big)\mathrm{d}s - \int_0^t\int_0^{+\infty} X_{s-}^i \mathbf{1}_{z\le\lambda(X_{s-}^i)}\mathcal{N}^i(\mathrm{d}s,\mathrm{d}z)$$

$$+ \frac{1}{N}\sum_{j\neq i}\int_0^t\int_0^{+\infty} \mathbf{1}_{z\le\lambda(X_{s-}^j)}\mathcal{N}^j(\mathrm{d}s,\mathrm{d}z).$$

Cut-off and mollifiers.

- Define $K_\varepsilon \to K$ as $\varepsilon \to 0$ where K_ε sufficiently nice to prove propagation of chaos with convergence speed $r_N(K_\varepsilon) \underset{N\to+\infty}{\longrightarrow} 0$ for a fixed $\varepsilon > 0$.
- Use a sequence $\varepsilon_N \underset{N\to+\infty}{\longrightarrow} 0$ depending on N.
- Try to prove propagation of chaos such that $r_N(K_{\varepsilon_N}) \underset{N\to+\infty}{\longrightarrow} 0$.

Example 1. $K_\varepsilon(x) = \frac{x}{\max(|x|,\varepsilon)^d} \to x/|x|^d$.

[Carrillo, Choi, Salem, *Comm. Con. Math. 21*, 2019]

Example 2 (moderate interaction). $K_\varepsilon(x) = \varepsilon^{-d} K_0(x/\varepsilon) \to \delta_x$.

[Oelschläger, *Ze. Wa. Ve. Ge. 69*, 1985]. [Jourdain, Méléard, *Ann. IHP Pro. St. 34*, 1998]

Local Lipschitz and exponential moments.

[Bolley, Cañizo, Carrillo, *Ma. Mo. Me. App. Sc. 21*, 2011]

- K local Lipschitz with polynomial growth or order p.
- The exponential moments $\mathbb{E}[\mathrm{e}^{\kappa|X_t^i|^{p'}}]$ and $\mathbb{E}[\mathrm{e}^{\kappa|\overline{X}_t^i|^{p'}}]$ are bounded on $(0,T)$ for some $\kappa > 0$ and $p' \geq p$.
- $\int |K(y-x)|^2 f_t(\mathrm{d}x) f_t(\mathrm{d}y) < +\infty$.

Entropy methods

For two probability measures $\mu, \nu \in \mathcal{P}(\mathscr{E})$, the relative entropy is defined by

$$H(\nu|\mu) := \int_{\mathscr{E}} \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \mathrm{d}\mu.$$

Lemma: chaos from entropy bounds

For any $k \leq N$, $f^N \in \mathcal{P}(E^N)$ and $f \in \mathcal{P}(E)$,

$$\frac{1}{2}\|f^{k,N} - f^{\otimes k}\|_{\mathrm{TV}}^2 \leq H(f^{k,N}|f^{\otimes k}) \leq \frac{k}{N} H(f^N|f^{\otimes N}).$$

[Ben Arous, Zeitouni, *Ann. IHP Prob. Sta. 35*, 1999]

[Ben Arous, Brunaud, *Sto. and Sto. Rep. 31*, 1990]

Lemma: bounding the entropy

Let $\mathrm{d}X_t^i = b(X_t^i, \mu_{\mathcal{X}_t^N})\mathrm{d}t + \sigma \mathrm{d}B_t^i$ be a **McKean-Vlasov process**,

$$H(f_I^N|f_I^{\otimes N}) = \frac{N}{2}\mathbb{E}\left[\int_0^T |b(X_t^1, \mu_{\mathcal{X}_t^N}) - b(X_t^1, f_t)|^2 \mathrm{d}t\right].$$

Key idea: Girsanov theorem

Entropy methods

- With the global Lipschitz bounded assumption of McKean's theorem, this is a **strengthening** result from Wasserstein to Total Variation convergence.
  [Malrieu, *Sto. Pro. App. 95*, 2001]
- **No regularity assumption** on b (*only* the well-posedness of the limit system).
  For linear interactions $b(x,\mu) = K \star \mu(x)$ with interaction kernel K:
  Bounded forces: $K \in L^\infty$ [Jabin, Wang, *J. Fun. An. 271*, 2016],
  Less than bounded $K \in W^{-1,\infty}$: [Jabin, Wang, *Inv. Math. 214*, 2018]
  Singular gradient systems $K = -\nabla W$: [Bresch, Jabin, Wang, *Duke Math. Journal*, 2022], [Serfaty, *ICM 2018*], [Duerinckx, *SIAM J. Ma. An. 48*, 2016]...
  Stochastic version: [Jabir, *arXiv:1907.09096*, 2019]
  $\longrightarrow$ Change of measure argument: bound observables for f_t instead of f_t^N.
- **No assumption on the form** of b [Lacker, *Elec. Com. Pro. 23*, 2018].
- Hierarchy of marginal entropies, quantitative chaos with **optimal rate** $\mathcal{O}(k/N)$.
  [Lacker, *arXiv:2105.02983*, 2021]

Entropy methods for jump and Boltzmann-Kac models via Girsanov transform?
[Léonard, *Séminaire de Probabilités XLIV*, 2012]

Beyond the classical theory

1. Some extensions
2. Some applications in numerical analysis, data science and optimization

Some extensions (check the program!)

- **Changing the noise...**

$$\mathrm{d}X_t^i = b\big(X_t^i, \mu_{\mathcal{X}_t^N}\big)\mathrm{d}t + \mathrm{d}M_t^i + \mathrm{d}B_t, \quad X_T^i = \xi^i.$$

  - **Individual noise:** M_t^i martingale measure, α-stable Lévy driven noise, terminal condition ξ^i...
  - **Environmental noise** B_t: SPDE limit, conditional propagation of chaos.

- **Changing the interactions...**

$$\mathrm{d}X_t^i = \frac{1}{N}\sum_{i=1}^{N} \Gamma_{ij}\, b(X_t^i, X_t^j, \mu_{\mathcal{X}_t^N}, \alpha_t^i)\mathrm{d}t + \mathrm{d}B_t^i.$$

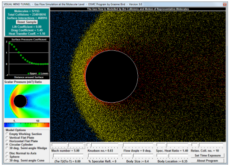
  - **Non-exchangeable systems:** (random) graph interactions $(\Gamma_{ij})_{ij}$, non-metric interactions ("topological") with K-nearest neighbors...
  - **Control** process α_t^i maximizing $J^i(\alpha^1, \ldots, \alpha^N)$.

- **Changing the scaling...**
  - **Boltzmann-Grad** scaling: binary interactions in a dilute regime.
  - **Diffusion scaling:** $1/\sqrt{N}$ instead of $1/N$.
  - **Fluctuation** process: $\eta_t^N = \sqrt{N}\big(\mu_{\mathcal{X}_t^N} - f_t\big)$.
  - Measure-valued limit: e.g. Fleming-Viot process

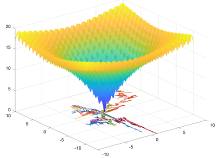
Beyond the classical theory

1. Some extensions

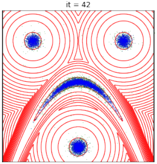
2. Some applications for the numerical analysis of PDE, data science and optimization



[Bird, *DSMC algorithm*, 1970]



[Totzeck, *Active Particles 3*, 2021]



[Clarté, D., Feydy, *EJS 16*, 2022]

Some applications in PDE, data science and optimization

- **Particle methods for nonlinear PDEs:** construct $X_t^1, \dots, X_t^N$ such that

$$\text{(some functional of) } \mu_{\mathcal{X}_t^N} \approx f_t,$$

  where f_t is the solution of a complicated PDE (e.g. Boltzmann, Burgers, vortex, Landau...).

- **Particle swarm optimization:** construct $X_t^1, \dots X_t^N$ such that

$$X_t^1, \dots X_t^N \underset{t \to +\infty}{\longrightarrow} x^\star$$

  where $x^\star$ is the minimizer of an objective function G.

- **MCMC sampling:** construct $X_t^1, \dots, X_t^N$ such that as $t \to +\infty$,

$$(X_t^1, \dots, X_t^N) \sim \pi^{\otimes N},$$

  where π is a probability density known up to a multiplicative constant.

- **Neural networks:** construct $\theta^1, \dots, \theta^N$ which minimize the risk functional

$$R(\boldsymbol{\theta}^N) := \sum_\ell \mathrm{Loss}\left(Y^\ell, \frac{1}{N}\sum_{i=1}^N \sigma(X^\ell, \theta^i)\right),$$

  where (X^ℓ, Y^ℓ) are some labelled data and σ is an activation function.

Simulating mean-field particle systems, final advertisment...

$\longrightarrow$ In all the previous applications, a **critical** limitation comes from the **high-computational cost** $\mathcal{O}(N^2)$ of discrete convolutions:

$$\text{Compute } \; y_i = \sum_{j=1}^{N} K(x_i, x_j) \; \text{ for } \; i \in \{1, \dots, N\}$$

- **Verlet list methods** in MD simulations for short-range interactions.
  [Leimkuhler, Matthews, *Molecular Dynamics*, 2015]
- **Super particles and tree methods** for long-range interactions.
  [Rokhlin, *J. Comp. Phys. 60*, 1985]
- "Approximate" K using so-called **kernel methods.**
  [Yang et al., *NeurIPS 25*, 2012]
- Randomly subsample the interactions via **random batch methods.**
  [Jin, Li, Liu, *J. Comp. Phys. 400*, 2020]
- Massively parallelized symbolic computations using **GPU routines.**

KeOps

[Charlier, Feydy, Glaunès, Collin, Durif, *J. Mach. Lea. Res. 22*, 2021]

Thank you for your attention!

References (lecture notes and review articles).

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[Sznitman, *Lecture at St Flour*, 1989]

[Méléard, in *Probabilistic Models for Nonlinear Partial Differential Equations*, 1996]

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[Golse, *Lecture notes. arXiv:1301.5494*, 2016]

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