Multidimensional indefinite stochastic Riccati equations and zero-sum linear-quadratic stochastic differential games with non-Markovian regime switching

Zuo Quan XU

The Hong Kong Polytechnic University

Based on joint work with Panpan Zhang (Shandong U)

A backward stochastic excursion with Ying Hu

17-19 June 2024

Introduction

Stochastic LQ control

- Wonham (SICON 1968). On a matrix Riccati equation of stochastic control.

Stochastic LQ control

- Wonham (SICON 1968). On a matrix Riccati equation of stochastic control.
- Bismut (SICON 1976). Linear quadratic optimal stochastic control with random coefficients.

Stochastic LQ control

- Wonham (SICON 1968). On a matrix Riccati equation of stochastic control.
- Bismut (SICON 1976). Linear quadratic optimal stochastic control with random coefficients.
- Kohlmann and Tang (SPA 2002). Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging.

Stochastic LQ control

- Wonham (SICON 1968). On a matrix Riccati equation of stochastic control.
- Bismut (SICON 1976). Linear quadratic optimal stochastic control with random coefficients.
- Kohlmann and Tang (SPA 2002). Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging.
- Tang (SICON 2003). General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations.

Stochastic LQ control

- Hu and Zhou (SICON 2005). Constrained stochastic LQ control with random coefficients, and application to mean-variance portfolio selection.

Stochastic LQ control

- Hu and Zhou (SICON 2005). Constrained stochastic LQ control with random coefficients, and application to mean-variance portfolio selection.
- Hu and Peng (C. R. Acad. Sci. Paris, Ser. 2006). On the comparison theorem for multidimensional BSDEs.

Stochastic LQ control

- Hu and Zhou (SICON 2005). Constrained stochastic LQ control with random coefficients, and application to mean-variance portfolio selection.
- Hu and Peng (C. R. Acad. Sci. Paris, Ser. 2006). On the comparison theorem for multidimensional BSDEs.
- Hu and Tang (SPA 2016). Multi-dimensional backward stochastic differential equations of diagonally quadratic generators.

Stochastic LQ control

- Hu and Zhou (SICON 2005). Constrained stochastic LQ control with random coefficients, and application to mean-variance portfolio selection.
- Hu and Peng (C. R. Acad. Sci. Paris, Ser. 2006). On the comparison theorem for multidimensional BSDEs.
- Hu and Tang (SPA 2016). Multi-dimensional backward stochastic differential equations of diagonally quadratic generators.
- Zhou and Li (AMO 2000). Continuous time mean-variance portfolio selection: a stochastic LQ framework.

Stochastic LQ control

- Hu and Zhou (SICON 2005). Constrained stochastic LQ control with random coefficients, and application to mean-variance portfolio selection.
- Hu and Peng (C. R. Acad. Sci. Paris, Ser. 2006). On the comparison theorem for multidimensional BSDEs.
- Hu and Tang (SPA 2016). Multi-dimensional backward stochastic differential equations of diagonally quadratic generators.
- Zhou and Li (AMO 2000). Continuous time mean-variance portfolio selection: a stochastic LQ framework.
- Zhou and Yin (SICON 2003). Markowitz's mean-variance portfolio selection with regime switching: A continuous-time model.

Our study

- Hu, Shi and Xu (AAP 2022). Constrained stochastic LQ control with regime switching and application to portfolio selection.

Our study

- Hu, Shi and Xu (AAP 2022). Constrained stochastic LQ control with regime switching and application to portfolio selection.
- HSX (ESAIM: COCV 2022). Constrained stochastic LQ control on infinite time horizon with regime switching.

Our study

- Hu, Shi and Xu (AAP 2022). Constrained stochastic LQ control with regime switching and application to portfolio selection.
- HSX (ESAIM: COCV 2022). Constrained stochastic LQ control on infinite time horizon with regime switching.
- HSX (MCRF 2023). Stochastic linear-quadratic control with a jump and regime switching on a random horizon.

Our study

- Hu, Shi and Xu (AAP 2022). Constrained stochastic LQ control with regime switching and application to portfolio selection.
- HSX (ESAIM: COCV 2022). Constrained stochastic LQ control on infinite time horizon with regime switching.
- HSX (MCRF 2023). Stochastic linear-quadratic control with a jump and regime switching on a random horizon.
- HSX (SIFIN 2023). Constrained monotone mean-variance problem with random coefficients.

Our study

- HSX (MCRF 2024). Inhomogeneous stochastic LQ control with regime switching and random coefficients.

Our study

- HSX (MCRF 2024). Inhomogeneous stochastic LQ control with regime switching and random coefficients.
- HSX (arXiv:2211.05291). Optimal consumption-investment with coupled constraints on consumption and investment strategies in a regime switching market with random coefficients.

Our study

- HSX (MCRF 2024). Inhomogeneous stochastic LQ control with regime switching and random coefficients.
- HSX (arXiv:2211.05291). Optimal consumption-investment with coupled constraints on consumption and investment strategies in a regime switching market with random coefficients.
- HSX (arXiv:2311.06512). Comparison theorems for multidimensional BSDEs with jumps and stochastic linear-quadratic control with jumps.

Stochastic differential games

- Isaacs (Differential games 1965): Deterministic framework.

Stochastic differential games

- Isaacs (Differential games 1965): Deterministic framework.
- Elliott and Kalton (Mem. AMS 1972). The existence of value in differential games.

Stochastic differential games

- Isaacs (Differential games 1965): Deterministic framework.
- Elliott and Kalton (Mem. AMS 1972). The existence of value in differential games.
- Fleming and Souganidis (Indiana Univ. Math. J. 1989). On the existence of value functions of two-player, zero-sum stochastic differential games.

Stochastic differential games

- Isaacs (Differential games 1965): Deterministic framework.
- Elliott and Kalton (Mem. AMS 1972). The existence of value in differential games.
- Fleming and Souganidis (Indiana Univ. Math. J. 1989). On the existence of value functions of two-player, zero-sum stochastic differential games.
- Buckdahn and Li (SICON 2008). Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations.

Stochastic differential games

- Isaacs (Differential games 1965): Deterministic framework.
- Elliott and Kalton (Mem. AMS 1972). The existence of value in differential games.
- Fleming and Souganidis (Indiana Univ. Math. J. 1989). On the existence of value functions of two-player, zero-sum stochastic differential games.
- Buckdahn and Li (SICON 2008). Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations.
- Yu (SICON 2015). An optimal feedback control-strategy pair for zero-sum linear-quadratic stochastic differential game: The Riccati equation approach.

Stochastic differential games

- Moon (IEEE TAC 2019). A sufficient condition for linear-quadratic stochastic zero-sum differential games for Markov jump systems.

Stochastic differential games

- Moon (IEEE TAC 2019). A sufficient condition for linear-quadratic stochastic zero-sum differential games for Markov jump systems.
- all the coefficients in the above games are deterministic, rendering the Riccati equation as 1-d ODEs.

Stochastic differential games

- Moon (IEEE TAC 2019). A sufficient condition for linear-quadratic stochastic zero-sum differential games for Markov jump systems.
- all the coefficients in the above games are deterministic, rendering the Riccati equation as 1-d ODEs.
- Lv (Automatica 2020). Two-player zero-sum stochastic differential games with regime switching: m-d ODE.

Stochastic differential games

- Moon (IEEE TAC 2019). A sufficient condition for linear-quadratic stochastic zero-sum differential games for Markov jump systems.
- all the coefficients in the above games are deterministic, rendering the Riccati equation as 1-d ODEs.
- Lv (Automatica 2020). Two-player zero-sum stochastic differential games with regime switching: m-d ODE.
- Moon (IEEE CSL 2020). A feedback Nash equilibrium for affine-quadratic zero-sum stochastic differential games with random coefficients.

Stochastic differential games

- Moon (IEEE TAC 2019). A sufficient condition for linear-quadratic stochastic zero-sum differential games for Markov jump systems.
- all the coefficients in the above games are deterministic, rendering the Riccati equation as 1-d ODEs.
- Lv (Automatica 2020). Two-player zero-sum stochastic differential games with regime switching: m-d ODE.
- Moon (IEEE CSL 2020). A feedback Nash equilibrium for affine-quadratic zero-sum stochastic differential games with random coefficients.
- This talk based on Zhang and Xu (arXiv:2309.05003).

Unconstrained inhomogeneous game

Features

- LQ+regime switching+random coefficients.
- One-dimensional state process (not essential for unconstrained problem, but essential for constrained problem).

Features

- LQ+regime switching+random coefficients.
- One-dimensional state process (not essential for unconstrained problem, but essential for constrained problem).
- Major difficulties:
  - The SREs are multidimensional and strong indefinite.

Features

- LQ+regime switching+random coefficients.
- One-dimensional state process (not essential for unconstrained problem, but essential for constrained problem).
- Major difficulties:
  - The SREs are multidimensional and strong indefinite.
  - The Riccati equations violate the standard Lipschitz and quadratic growth conditions.

Features

- LQ+regime switching+random coefficients.
- One-dimensional state process (not essential for unconstrained problem, but essential for constrained problem).
- Major difficulties:
  - The SREs are multidimensional and strong indefinite.
  - The Riccati equations violate the standard Lipschitz and quadratic growth conditions.
  - Linear BSDEs with unbounded coefficients for inhomogeneous game.

Main tools used

- Existence of BSDEs with Lipschitz generators: Tang (SICON 2003).

Main tools used

- Existence of BSDEs with Lipschitz generators: Tang (SICON 2003).
- Approximate from above and below: Fan, Hu and Tang (JDE 2023).

Main tools used

- Existence of BSDEs with Lipschitz generators: Tang (SICON 2003).
- Approximate from above and below: Fan, Hu and Tang (JDE 2023).
- Comparison theorem for multi-dimensional BSDEs: Hu and Peng (C. R. Acad. Sci. Paris, Ser. 2006).

Main tools used

- Existence of BSDEs with Lipschitz generators: Tang (SICON 2003).
- Approximate from above and below: Fan, Hu and Tang (JDE 2023).
- Comparison theorem for multi-dimensional BSDEs: Hu and Peng (C. R. Acad. Sci. Paris, Ser. 2006).
- Existence of limit: Kobylanski (AP 2000). Backward stochastic differential equations and partial differential equations with quadratic growth, Cvitanic and Zhang (Contract Theory in Continuous-Time Models 2012).

Market

- A complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- A Brownian motion $W(t) = (W_1(t), \ldots, W_n(t))'$.
- A continuous-time stationary Markov chain α_t valued in a finite state space $\mathcal{M} = \{1, 2, \ldots, \ell\}$ with $\ell > 1$.
- The Markov chain α_t has a generator $Q = (q_{ij})_{\ell\times\ell}$ with $q_{ij} \geqslant 0$ for $i \neq j$ and $\sum_{j=1}^{\ell} q_{ij} = 0$ for every $i \in \mathcal{M}$.
- $W(t)$ and α_t are independent.
- Filtrations: $\mathcal{F}_t = \sigma\{W(s), \alpha_s : 0 \leqslant s \leqslant t\} \bigvee \mathcal{N}$ and $\mathcal{F}_t^W = \sigma\{W(s) : 0 \leqslant s \leqslant t\} \bigvee \mathcal{N}$.

State process

- An inhomogeneous scalar-valued controlled system:

$$\begin{cases} \mathrm{d}X(t) = \big[A(t,\alpha_t)X(t) + b(t,\alpha_t) \\ \qquad\qquad + B_1(t,\alpha_t)^\top u_1(t) + B_2(t,\alpha_t)^\top u_2(t)\big]\,\mathrm{d}t \\ \qquad\quad + \big[C(t,\alpha_t)X(t) + \sigma(t,\alpha_t) \\ \qquad\qquad + D_1(t,\alpha_t)u_1(t) + D_2(t,\alpha_t)u_2(t)\big]^\top \mathrm{d}W(t), \\ X(0) = x \in \mathbb{R}, \quad \alpha_0 = i_0 \in \mathcal{M}, \end{cases} \tag{1}$$

where $A(t,i),\ B(t,i),\ C(t,i),\ D(t,i)$ are all $\{\mathcal{F}_t^W\}_{t\geqslant 0}$-adapted processes of suitable sizes for $i \in \mathcal{M}$.

Cost functional

- Quadratic cost functional

$$J_{x,i_0}(u_1, u_2) = \mathbb{E}\Big[\int_0^T \Big(K(t,\alpha_t)X(t)^2 + u_1(t)^\top R_{11}(t,\alpha_t)u_1(t) + u_2(t)^\top R_{22}(t,\alpha_t)u_2(t) + 2u_1(t)^\top R_{12}(t,\alpha_t)u_2(t) \Big)\,\mathrm{d}t + G(\alpha_T)X(T)^2 \Big]. \tag{2}$$

where $K(t,i)$ and $R(t,i)$ are $\{\mathcal{F}_t^W\}_{t\geqslant 0}$-adapted processes, $G(i)$ are $\mathcal{F}_T^W$-measurable random variables.

- For $k \in \{1,2\}$, the control process $u_k(\cdot)$ of Player k is chosen from the admissible control set $\mathcal{U}_k = L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_k})$.

Admissible strategies

Following Buckdahn and Li (SICON 2008), Yu (SICON 2015), Lv (Automatica 2020).

Definition 1 (Admissible strategies)

An admissible strategy for Player 1 is a mapping $\beta_1 : \mathcal{U}_2 \to \mathcal{U}_1$ such that for any $\mathcal{F}_t$-stopping time $\tau : \Omega \to [0, T]$ and any two controls u_2, $\overline{u}_2 \in \mathcal{U}_2$ with $u_2 = \overline{u}_2$ on $[0, \tau]$, it holds that $\beta_1(u_2) = \beta_1(\overline{u}_2)$ on $[0, \tau]$. The set of all admissible strategies for Player 1 is denoted by $\mathcal{A}_1$. Admissible strategies $\beta_2 : \mathcal{U}_1 \to \mathcal{U}_2$ and the collection $\mathcal{A}_2$ of them for Player 2 are defined similarly.

Players' values

- For $(x, i_0) \in \mathbb{R} \times \mathcal{M}$, we define Player 1's value as

$$V_1(x, i_0) \triangleq \inf_{\beta_2 \in \mathcal{A}_2} \sup_{u_1 \in \mathcal{U}_1} J_{x,i_0}\big(u_1, \beta_2(u_1)\big),$$

and Player 2's value as

$$V_2(x, i_0) \triangleq \sup_{\beta_1 \in \mathcal{A}_1} \inf_{u_2 \in \mathcal{U}_2} J_{x,i_0}\big(\beta_1(u_2), u_2\big).$$

Optimal pairs

- When $V_1(x, i_0)$ is finite, the zero-sum LQ stochastic differential game for Player 1 is to find an optimal pair $(u_1^*, \beta_2^*) \in \mathcal{U}_1 \times \mathcal{A}_2$ such that

$$J_{x,i_0}(u_1^*, \beta_2^*(u_1^*)) = V_1(x, i_0).$$

- If the two players' values are equal, we call the common value the value of the game.

Admissible feedback control for for Player 1

Definition 2

An admissible feedback control for Player 1 is a mapping $\pi : [0, T] \times \Omega \times \mathcal{M} \times \mathbb{R} \to \mathbb{R}^{m_1}$, such that

1. $\pi(\cdot, \cdot, i, x)$ is $\mathcal{F}_t^W$-adapted;
2. for each $u_2(\cdot) \in \mathcal{U}_2$, there is a unique solution $X(\cdot)$ to

$$\begin{cases} \mathrm{d}X(t) = \left[AX(t) + B_1^\top \pi(t, \alpha_t, X(t)) + B_2^\top u_2(t) + b\right] \mathrm{d}t \\ \qquad + \left[CX(t) + D_1 \pi(t, \alpha_t, X(t)) + D_2 u_2(t) + \sigma\right]^\top \mathrm{d}W(t), \\ X(0) = x \in \mathbb{R}, \quad \alpha_0 = i_0 \in \mathcal{M}, \end{cases} \tag{3}$$

and $\pi(\cdot, \alpha_\cdot, X(\cdot)) \in \mathcal{U}_1$.

Admissible feedback strategy for Player 2

Definition 3

An admissible feedback strategy for Player 2 is a mapping $\Pi : [0, T] \times \Omega \times \mathcal{M} \times \mathbb{R} \times \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$, such that

1. $\Pi(\cdot, \cdot, i, x, u)$ is $\mathcal{F}_t^W$-adapted;
2. for each $u_1(\cdot) \in \mathcal{U}_1$, there exists a unique solution $X(\cdot)$ to the following SDE:

$$\begin{cases} \mathrm{d}X(t) = \big[AX(t) + B_1^\top u_1(t) + B_2^\top \Pi(t, \alpha_t, X(t), u_1(t)) + b\big]\,\mathrm{d}t \\ \qquad\qquad + \big[CX(t) + D_1 u_1(t) + D_2 \Pi(t, \alpha_t, X(t), u_1(t)) + \sigma\big]^\top \mathrm{d}W(t), \\ X(0) = x \in \mathbb{R}, \quad \alpha_0 = i_0 \in \mathcal{M}, \end{cases}$$

   and $\beta_2 : u_1(\cdot) \mapsto \Pi(\cdot, \alpha_\cdot, X(\cdot), u_1(\cdot)) \in \mathcal{A}_2$.

Optimal feedback control-strategy pair for Player 1's value

Definition 4

A pair (π, Π) is called an optimal feedback control-strategy pair for Player 1's value if the pair (u_1, β_2) is optimal, that is,

$$J_{x,i_0}(u_1, \beta_2(u_1)) = V_1(x, i_0),$$

where (u_1, β_2) is defined by $u_1(\cdot) = \pi(\cdot, \alpha_\cdot, X(\cdot))$, $\beta_2 : u_1(\cdot) \mapsto \Pi(\cdot, \alpha_\cdot, X(\cdot), u_1(\cdot))$, and $X(\cdot)$ satisfies

$$\begin{cases} \mathrm{d}X(t) = \left[AX(t) + B_1^\top \pi(t, \alpha_t, X(t)) + B_2^\top \Pi(t, \alpha_t, X(t), u_1(t)) + b\right] \mathrm{d}t \\ \qquad + \left[CX(t) + D_1 \pi(t, \alpha_t, X(t)) + D_2 \Pi(t, \alpha_t, X(t), u_1(t)) + \sigma\right]^\top \mathrm{d}W(t), \\ X(0) = x \in \mathbb{R}, \quad \alpha_0 = i_0 \in \mathcal{M}. \end{cases}$$

Multidimensional indefinite SRE for unconstrained inhomogeneous game

Definition

Define

$$\widehat{R}(t,i,P) \triangleq R(t,i) + PD(t,i)^\top D(t,i),$$
$$\widehat{C}(t,i,P,\Lambda) \triangleq PB(t,i) + D(t,i)^\top [PC(t,i) + \Lambda],$$
$$\hat{\sigma}(t,i,P,\varphi,\Delta) \triangleq \varphi B(t,i) + D(t,i)^\top [P\sigma(t,i) + \Delta],$$

and

$$H_1(t,i,P,\Lambda) \triangleq -\widehat{C}^\top \widehat{R}^{-1} \widehat{C},$$
$$H_2(t,i,P,\Lambda,\varphi,\Delta) \triangleq -\widehat{C}^\top \widehat{R}^{-1} \hat{\sigma},$$
$$H_3(t,i,P,\varphi,\Delta) \triangleq -\hat{\sigma}^\top \widehat{R}^{-1} \hat{\sigma}.$$

Definition

An indefinite SRE

$$\begin{cases} \mathrm{d}P(t,i) = -\Big[K(t,i) + P(t,i)\big[2A(t,i) + C(t,i)^\top C(t,i)\big] + 2C(t,i)^\top \Lambda(t,i) \\ \qquad\qquad + H_1(t,i,P(t,i),\Lambda(t,i)) + \sum_{j\in\mathcal{M}} q_{ij}P(t,j)\Big]\mathrm{d}t \\ \qquad\qquad + \Lambda(t,i)^\top \mathrm{d}W(t), \\ P(T,i) = G(i),\ P(\cdot,i)\in[-\bar{\epsilon},\bar{\epsilon}],\ \text{ for all } i\in\mathcal{M}, \end{cases} \tag{4}$$

and a linear BSDE (solved in Hu, Shi and Xu (MRCF 2024))

$$\begin{cases} \mathrm{d}\varphi(t,i) = -\Big[P(t,i)\big[b(t,i) + C(t,i)^\top\sigma(t,i)\big] + \sigma(t,i)^\top\Lambda(t,i) + A(t,i)\varphi(t,i) \\ \qquad\qquad + C(t,i)^\top\Delta(t,i) + H_2(t,i,P(t,i),\Lambda(t,i),\varphi(t,i),\Delta(t,i)) \\ \qquad\qquad + \sum_{j\in\mathcal{M}} q_{ij}\varphi(t,j)\Big]\mathrm{d}t + \Delta(t,i)^\top\mathrm{d}W(t), \\ \varphi(T,i) = 0,\ \text{ for all } i\in\mathcal{M}. \end{cases} \tag{5}$$

Existence

On the solvability of indefinite SREs or indefinite quadratic BSDEs:

- Chen, Li, and Zhou (SICON 1998). Stochastic linear quadratic regulators with indefinite control weight costs.

Existence

On the solvability of indefinite SREs or indefinite quadratic BSDEs:

- Chen, Li, and Zhou (SICON 1998). Stochastic linear quadratic regulators with indefinite control weight costs.
- Hu and Zhou (SICON 2003). Indefinite stochastic Riccati equations.

Existence

On the solvability of indefinite SREs or indefinite quadratic BSDEs:

- Chen, Li, and Zhou (SICON 1998). Stochastic linear quadratic regulators with indefinite control weight costs.
- Hu and Zhou (SICON 2003). Indefinite stochastic Riccati equations.
- Qian and Zhou (SICON 2013). Existence of solutions to a class of indefinite stochastic Riccati equations.

Existence

On the solvability of indefinite SREs or indefinite quadratic BSDEs:

- Chen, Li, and Zhou (SICON 1998). Stochastic linear quadratic regulators with indefinite control weight costs.
- Hu and Zhou (SICON 2003). Indefinite stochastic Riccati equations.
- Qian and Zhou (SICON 2013). Existence of solutions to a class of indefinite stochastic Riccati equations.
- Du (SICON 2015). Solvability conditions for indefinite linear quadratic optimal stochastic control problems and associated stochastic Riccati equations.

Existence

On the solvability of indefinite SREs or indefinite quadratic BSDEs:

- Chen, Li, and Zhou (SICON 1998). Stochastic linear quadratic regulators with indefinite control weight costs.
- Hu and Zhou (SICON 2003). Indefinite stochastic Riccati equations.
- Qian and Zhou (SICON 2013). Existence of solutions to a class of indefinite stochastic Riccati equations.
- Du (SICON 2015). Solvability conditions for indefinite linear quadratic optimal stochastic control problems and associated stochastic Riccati equations.
- Hu, Shi and Xu (AAP 2022). Constrained stochastic LQ control with regime switching and application to portfolio selection.

Existence

Assumptions on the coefficients.

- All the coefficients are bounded.

Existence

Assumptions on the coefficients.

- All the coefficients are bounded.
- The matrices $D_1^\top D_1$ and $D_2^\top D_2$ are uniformly positive.

Existence

Assumptions on the coefficients.

- All the coefficients are bounded.
- The matrices $D_1^\top D_1$ and $D_2^\top D_2$ are uniformly positive.
- $R_{11} \leqslant -(\epsilon + \bar{\epsilon}\bar{c}_2)I_{m_1}$ and $R_{22} \geqslant (\epsilon + \bar{\epsilon}\bar{c}_2)I_{m_2}$.

Existence

Assumptions on the coefficients.

- All the coefficients are bounded.
- The matrices $D_1^\top D_1$ and $D_2^\top D_2$ are uniformly positive.
- $R_{11} \leqslant -(\epsilon + \bar{\epsilon}\bar{c}_2)I_{m_1}$ and $R_{22} \geqslant (\epsilon + \bar{\epsilon}\bar{c}_2)I_{m_2}$.
- No other restrictions on the signs of K, G and R_{12} in the cost functional

$$J_{x,i_0}(u_1,u_2) = \mathbb{E}\bigg[\int_0^T \Big(K(t,\alpha_t)X(t)^2 + u_1(t)^\top R_{11}(t,\alpha_t)u_1(t) + u_2(t)^\top R_{22}(t,\alpha_t)u_2(t) + 2u_1(t)^\top R_{12}(t,\alpha_t)u_2(t)\Big)\,\mathrm{d}t + G(\alpha_T)X(T)^2\bigg].$$

Existence

Suppose $P \in [-\bar{\epsilon}, \bar{\epsilon}]$.

- Since $\widehat{R}_{11} \leqslant -\epsilon I_{m_1}$ and $\widehat{R}_{22} \geqslant \epsilon I_{m_2}$, $\widehat{R}$ is strongly indefinite.

Existence

Suppose $P \in [-\bar{\epsilon}, \bar{\epsilon}]$.

- Since $\widehat{R}_{11} \leqslant -\epsilon I_{m_1}$ and $\widehat{R}_{22} \geqslant \epsilon I_{m_2}$, $\widehat{R}$ is strongly indefinite.
- There is no upper or lower bounds on the eigenvalues of $\widehat{R}$.

Existence

Suppose $P \in [-\bar{\epsilon}, \bar{\epsilon}]$.

- Since $\widehat{R}_{11} \leqslant -\epsilon I_{m_1}$ and $\widehat{R}_{22} \geqslant \epsilon I_{m_2}$, $\widehat{R}$ is strongly indefinite.
- There is no upper or lower bounds on the eigenvalues of $\widehat{R}$.
- It holds that

$$|H_1| \leqslant \frac{2(c_3\bar{\epsilon}^2 + \bar{c}_2|\Lambda|^2)}{\epsilon}.$$

  This only holds for $P \in [-\bar{\epsilon}, \bar{\epsilon}]$.

Existence

Suppose $P \in [-\bar{\epsilon}, \bar{\epsilon}]$.

- Since $\widehat{R}_{11} \leqslant -\epsilon I_{m_1}$ and $\widehat{R}_{22} \geqslant \epsilon I_{m_2}$, $\widehat{R}$ is strongly indefinite.
- There is no upper or lower bounds on the eigenvalues of $\widehat{R}$.
- It holds that

$$|H_1| \leqslant \frac{2(c_3\bar{\epsilon}^2 + \bar{c}_2|\Lambda|^2)}{\epsilon}.$$

  This only holds for $P \in [-\bar{\epsilon}, \bar{\epsilon}]$. Globally, it violates the Lipchitz and quadratic growth conditions.

- In the existing control problems, it is often $H_1 \geqslant 0$.

Existence

- For $P \in [-\bar{\epsilon}, \bar{\epsilon}]$, consider decomposition

$$\begin{aligned} H_1 &= -\big[\widehat{C}_1 - \widehat{R}_{12}\widehat{R}_{22}^{-1}\widehat{C}_2\big]^\top \widetilde{R}_{11}^{-1}\big[\widehat{C}_1 - \widehat{R}_{12}\widehat{R}_{22}^{-1}\widehat{C}_2\big] \\ &\quad - \widehat{C}_2^\top \widehat{R}_{22}^{-1}\widehat{C}_2 \\ &= \overline{H}_1 + \underline{H}_1. \end{aligned}$$

Existence

- For $P \in [-\bar{\epsilon}, \bar{\epsilon}]$, consider decomposition

$$\begin{aligned} H_1 &= -\big[\widehat{C}_1 - \widehat{R}_{12}\widehat{R}_{22}^{-1}\widehat{C}_2\big]^\top \widetilde{R}_{11}^{-1}\big[\widehat{C}_1 - \widehat{R}_{12}\widehat{R}_{22}^{-1}\widehat{C}_2\big] \\ &\quad - \widehat{C}_2^\top \widehat{R}_{22}^{-1}\widehat{C}_2 \\ &= \overline{H}_1 + \underline{H}_1. \end{aligned}$$

- When $P \in [-\bar{\epsilon}, \bar{\epsilon}]$, it holds that

$$0 \leqslant \overline{H}_1, -\underline{H}_1 \leqslant \frac{2(c_3\bar{\epsilon}^2 + \bar{c}_2|\Lambda|^2)}{\epsilon}.$$

Existence

- For $P \in [-\bar{\epsilon}, \bar{\epsilon}]$, consider decomposition

$$\begin{aligned} H_1 &= -\big[\widehat{C}_1 - \widehat{R}_{12}\widehat{R}_{22}^{-1}\widehat{C}_2\big]^\top \widetilde{R}_{11}^{-1}\big[\widehat{C}_1 - \widehat{R}_{12}\widehat{R}_{22}^{-1}\widehat{C}_2\big] \\ &\quad - \widehat{C}_2^\top \widehat{R}_{22}^{-1}\widehat{C}_2 \\ &= \overline{H}_1 + \underline{H}_1. \end{aligned}$$

- When $P \in [-\bar{\epsilon}, \bar{\epsilon}]$, it holds that

$$0 \leqslant \overline{H}_1, -\underline{H}_1 \leqslant \frac{2(c_3\bar{\epsilon}^2 + \bar{c}_2|\Lambda|^2)}{\epsilon}.$$

- Extend the definitions of $\overline{H}_1$ and $\underline{H}_1$ to all $P \in \mathbb{R}$ while keeping the above estimates hold.

Existence

- Let

$$\overline{H}_1^k(t,i,P,\Lambda) \triangleq \inf_{(\widetilde{P},\widetilde{\Lambda})\in\mathbb{R}\times\mathbb{R}^n} \Big\{\overline{H}_1(t,i,\widetilde{P},\widetilde{\Lambda}) + k|P-\widetilde{P}| + k|\Lambda-\widetilde{\Lambda}|\Big\},$$

$$\underline{H}_1^k(t,i,P,\Lambda) \triangleq \sup_{(\widetilde{P},\widetilde{\Lambda})\in\mathbb{R}\times\mathbb{R}^n} \Big\{\underline{H}_1(t,i,\widetilde{P},\widetilde{\Lambda}) - k|P-\widetilde{P}| - k|\Lambda-\widetilde{\Lambda}|\Big\}.$$

- Then $\overline{H}_1^k$ and $\underline{H}_1^k$ are uniformly Lipschitz.
- And $\overline{H}_1^k$ is increasing to $\overline{H}_1$ and $\underline{H}_1^k$ is decreasing to $\underline{H}_1$ as k goes to infinity.
- Also

$$0 \leqslant \overline{H}_1^k, -\underline{H}_1^k \leqslant \frac{2(c_3\bar{\epsilon}^2 + \bar{c}_2|\Lambda|^2)}{\epsilon}.$$

Existence

- The following multidimensional BSDE has a Lipschitz generator, so it is solvable

$$\begin{cases} \mathrm{d}P^{k,\bar{k}}(t,i) = -\big[g(t,i,\boldsymbol{P}^{k,\bar{k}}(t),\Lambda^{k,\bar{k}}(t,i)) + \overline{H}_1^{\bar{k}}(t,i,P^{k,\bar{k}}(t,i),\Lambda^{k,\bar{k}}(t,i)) \\ \qquad\qquad + \underline{H}_1^{k}(t,i,P^{k,\bar{k}}(t,i),\Lambda^{k,\bar{k}}(t,i))\big]\,\mathrm{d}t + \Lambda^{k,\bar{k}}(t,i)^{\top}\,\mathrm{d}W(t), \\ P^{k,\bar{k}}(T,i) = G(i), \text{ for all } i \in \mathcal{M}. \end{cases} \tag{6}$$

- Using monotonicity in k and $\bar{k}$ and Cvitanic and Zhang (Contract Theory in Continuous-Time Models 2012):

$$\begin{cases} \mathrm{d}P(t,i) = -\big[g(t,i,\boldsymbol{P}(t),\Lambda(t,i)) + \overline{H}_1(t,i,P(t,i),\Lambda(t,i)) \\ \qquad\qquad + \underline{H}_1(t,i,P(t,i),\Lambda(t,i))\big]\,\mathrm{d}t + \Lambda(t,i)^{\top}\,\mathrm{d}W(t), \\ P(T,i) = G(i), \text{ for all } i \in \mathcal{M}. \end{cases}$$

Existence

- Comparing to the BSDEs with generators

$$\underline{g}(t,i,\boldsymbol{P},\Lambda)=c_1\sum_{j=1}^{l}P_j+2C(t,i)^\top\Lambda-\overline{K}-\frac{2(c_3\bar{\epsilon}^2+c_2|\Lambda|^2)}{\epsilon},$$

and

$$\overline{g}(t,i,\boldsymbol{P},\Lambda)=c_1\sum_{j=1}^{l}P_j+2C(t,i)^\top\Lambda+\overline{K}+\frac{2(c_3\bar{\epsilon}^2+c_2|\Lambda|^2)}{\epsilon},$$

we get $-\bar{\epsilon}\leqslant P^{k,\bar{k}}(\cdot,i)\leqslant\bar{\epsilon}$, so $P(\cdot,i)\in[-\bar{\epsilon},\bar{\epsilon}]$.

- Existence is established since $H_1=\overline{H}_1+\underline{H}_1$ when $P\in[-\bar{\epsilon},\bar{\epsilon}]$.

Uniqueness

- The log transformation in Hu, Shi and Xu (AAP 2022) fails for our SRE.
- Since $\widehat{R}$ is indefinite, we cannot find a transformation such that the quadratic term in generator is monotone.
- In zero-sum game, the two players take opposite goals, so monotonicity is in general losing.
- We establish the uniqueness result via a verification theorem.
- It is challenging to establish the uniqueness result by pure BSDE methods.

Optimal feedback control-strategy pair for unconstrained inhomogeneous game

Definition

Let

$$\begin{cases} u_1^*(t,i,X(t)) = -\widetilde{R}_{11}^{-1}\big\{\big[\widehat{C}_1 - \widehat{R}_{12}\widehat{R}_{22}^{-1}\widehat{C}_2\big]X(t) + \hat{\sigma}_1 - \widehat{R}_{12}\widehat{R}_{22}^{-1}\hat{\sigma}_2\big\}, \\ \beta_2^*(t,i,u_1(t),X(t)) = -\widehat{R}_{22}^{-1}\big[\widehat{R}_{12}^\top u_1(t) + \widehat{C}_2 X(t) + \hat{\sigma}_2\big], \end{cases} \tag{7}$$

and

$$\begin{cases} u_2^*(t,i,X(t)) = -\widetilde{R}_{22}^{-1}\big\{\big[\widehat{C}_2 - \widehat{R}_{12}^\top\widehat{R}_{11}^{-1}\widehat{C}_1\big]X(t) + \hat{\sigma}_2 - \widehat{R}_{12}^\top\widehat{R}_{11}^{-1}\hat{\sigma}_1\big\}, \\ \beta_1^*(t,i,u_2(t),X(t)) = -\widehat{R}_{11}^{-1}\big[\widehat{R}_{12} u_2(t) + \widehat{C}_1 X(t) + \hat{\sigma}_1\big]. \end{cases} \tag{8}$$

The coefficients depends on Λ.

Lemma 5

The feedback control-strategy pair of Player 1 (resp., Player 2) defined by(7) *(resp.,*(8)*) is admissible.*

Optimality

Lemma 6

It holds that

$$J_{x,i_0}(u_1,u_2) = P(0,i_0)x^2 + 2\varphi(0,i_0)x + \mathbb{E}\Big[\int_0^T \Big(\big[u_2 - \beta_2^*(u_1)\big]^\top \widehat{R}_{22}\big[u_2 - \beta_2^*(u_1)\big] + (u_1 - u_1^*)^\top \widetilde{R}_{11}(u_1 - u_1^*) + P\sigma^\top\sigma + 2(\varphi b + \sigma^\top\Delta) + H_3\Big)\,\mathrm{d}s\Big]. \tag{9}$$

Consequently,

1. $J_{x,i_0}(u_1,\beta_2^*(u_1)) \leqslant J_{x,i_0}(u_1,\beta_2(u_1))$ *for any* $u_1 \in \mathcal{U}_1$ *and* $\beta_2 \in \mathcal{A}_2$. *Moreover, the equation holds if and only if* $\beta_2(u_1) = \beta_2^*(u_1)$;
2. $J_{x,i_0}(u_1^*,\beta_2^*(u_1^*)) \geqslant J_{x,i_0}(u_1,\beta_2^*(u_1))$ *for any* $u_1 \in \mathcal{U}_1$. *Moreover, the equation holds if and only if* $u_1 = u_1^*$.

Optimality

Theorem 7 (Solution for the unconstrained inhomogeneous game(1)-(2))

An optimal control-strategy pair for Player 1's (resp., Player 2's) value is given by(7) *(resp.,*(8)*). Moreover, the game has a value*

$$
\begin{aligned}
&V_1(x, i_0) = V_2(x, i_0) = \\
&= P(0, i_0)x^2 + 2\varphi(0, i_0)x + \int_0^T \mathbb{E}\Big[P(t, \alpha_t)\sigma(t, \alpha_t)^\top \sigma(t, \alpha_t) + 2\big[\varphi(t, \alpha_t)b(t, \alpha_t) \\
&\quad + \sigma(t, \alpha_t)^\top \Delta(t, \alpha_t)\big] + H_3(t, \alpha_t, P(t, \alpha_t), \varphi(t, \alpha_t), \Delta(t, \alpha_t))\Big]\, \mathrm{d}t,
\end{aligned}
$$

where $\big(P(\cdot, i), \Lambda(\cdot, i)\big)_{i \in \mathcal{M}}$ *and* $\big(\varphi(\cdot, i), \Delta(\cdot, i)\big)_{i \in \mathcal{M}}$ *are solutions of*(4)-(5)*.*

Constrained homogeneous game

Problem formulation

- Then state process(1) is a homogeneous system:

$$\begin{cases} \mathrm{d}X(t) = \big[A(t,\alpha_t)X(t) + B(t,\alpha_t)^\top u(t)\big]\,\mathrm{d}t \\ \qquad\qquad + \big[C(t,\alpha_t)X(t) + D(t,\alpha_t)u(t)\big]^\top \mathrm{d}W(t), \\ X(0) = x \in \mathbb{R}, \quad \alpha_0 = i_0 \in \mathcal{M}. \end{cases} \tag{10}$$

- The admissible control sets are defined as

$$\widetilde{\mathcal{U}}_k = \big\{u_k(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_k}) \;\big|\; u_k(\cdot) \in \Gamma_k\big\},\ k \in \{1,2\},$$

where $\Gamma_1 \in \mathbb{R}^{m_1}$, $\Gamma_2 \in \mathbb{R}^{m_2}$ are two closed convex cones.

Multidimensional indefinite SRE for constrained homogeneous game

Definition

Define

$$
\begin{aligned}
f_{1k}(t,i,P,\Lambda,v_2) &= \max_{v_1\in\Gamma_1}\left\{v_1^\top \widehat{R}_{11}v_1 + 2v_1^\top \widehat{R}_{12}v_2 - 2(-1)^k \widehat{C}_1^\top v_1\right\},\\
f_{2k}(t,i,P,\Lambda,v_1) &= \min_{v_2\in\Gamma_2}\left\{v_2^\top \widehat{R}_{22}v_2 + 2v_1^\top \widehat{R}_{12}v_2 - 2(-1)^k \widehat{C}_2^\top v_2\right\},\\
\widetilde{H}_{1k}(t,i,P,\Lambda) &= \max_{v_1\in\Gamma_1}\left\{v_1^\top \widehat{R}_{11}v_1 - 2(-1)^k \widehat{C}_1^\top v_1 + f_{2k}(t,i,P,\Lambda,v_1)\right\},\\
\widetilde{H}_{2k}(t,i,P,\Lambda) &= \min_{v_2\in\Gamma_2}\left\{v_2^\top \widehat{R}_{22}v_2 - 2(-1)^k \widehat{C}_2^\top v_2 + f_{1k}(t,i,P,\Lambda,v_2)\right\}.
\end{aligned}
$$

Definition

- We have, for any c sufficiently large,

$$\widetilde{H}_{1k}=\widetilde{H}_{2k}=\max_{\substack{v_1\in\Gamma_1\\ |v_1|\leqslant c(1+|\Lambda|)}}\ \min_{\substack{v_2\in\Gamma_2\\ |v_2|\leqslant c(1+|\Lambda|)}}\mathcal{H}_k=\min_{\substack{v_2\in\Gamma_2\\ |v_2|\leqslant c(1+|\Lambda|)}}\ \max_{\substack{v_1\in\Gamma_1\\ |v_1|\leqslant c(1+|\Lambda|)}}\mathcal{H}_k=\widetilde{H}_k,$$

and

$$\begin{aligned}\mathcal{H}_k \triangleq\ & v_1^\top \widehat{R}_{11}v_1-2(-1)^k\widehat{C}_1^\top v_1\\ &+v_2^\top\widehat{R}_{22}v_2+2v_1^\top\widehat{R}_{12}v_2-2(-1)^k\widehat{C}_2^\top v_2.\end{aligned}$$

- Because of constraint, the SRE for LQ game(10)-(2) is not a single BSDE, but consists of a decoupled pair of BSDEs:

$$\begin{cases}\mathrm{d}P_k(t,i)=-\Big[K(t,i)+P_k(t,i)\big[2A(t,i)+C(t,i)^\top C(t,i)\big]+2C(t,i)^\top\Lambda_k(t,i)\\ \qquad\qquad +\widetilde{H}_k(t,i,P_k(t,i),\Lambda_k(t,i))+\sum_{j\in\mathcal{M}}q_{ij}P_k(t,j)\Big]\,\mathrm{d}t+\Lambda_k(t,i)^\top\,\mathrm{d}W(t),\\ P_k(T,i)=G(i),\ P_k(\cdot,i)\in[-\bar{\epsilon},\bar{\epsilon}],\ \text{ for all } i\in\mathcal{M},\ k\in\{1,2\}.\end{cases}\tag{11}$$

Optimal feedback control-strategy pair for constrained homogeneous game

Definition

Let

$$\begin{aligned}
\hat{v}_{1k}(t,i,P,\Lambda) &\triangleq \arg\max_{v_1\in\Gamma_1}\left\{v_1^\top \widehat{R}_{11}v_1 - 2(-1)^k\widehat{C}_1^\top v_1 + f_{2k}(v_1)\right\},\\
\hat{v}_{2k}(t,i,P,\Lambda) &\triangleq \arg\min_{v_2\in\Gamma_2}\left\{v_2^\top \widehat{R}_{22}v_2 - 2(-1)^k\widehat{C}_2^\top v_2 + f_{1k}(v_2)\right\},\\
\hat{\beta}_{1k}(t,i,P,\Lambda,v_2) &\triangleq \arg\max_{v_1\in\Gamma_1}\left\{v_1^\top \widehat{R}_{11}v_1 + 2v_1^\top \widehat{R}_{12}v_2 - 2(-1)^k\widehat{C}_1^\top v_1\right\},\\
\hat{\beta}_{2k}(t,i,P,\Lambda,v_1) &\triangleq \arg\min_{v_2\in\Gamma_2}\left\{v_2^\top \widehat{R}_{22}v_2 + 2v_1^\top \widehat{R}_{12}v_2 - 2(-1)^k\widehat{C}_2^\top v_2\right\},
\end{aligned}$$

and

$$\begin{cases}
u_1^*(t,i,X(t)) = \hat{v}_{11}(P_1(t,i),\Lambda_1(t,i))X(t)^+ + \hat{v}_{12}(P_2(t,i),\Lambda_2(t,i))X(t)^-,\\
\beta_2^*(t,i,u_1(t),X(t)) = \hat{\beta}_{21}(P_1(t,i),\Lambda_1(t,i),v_1(t))X(t)^+\\
\qquad + \hat{\beta}_{22}(P_2(t,i),\Lambda_2(t,i),v_1(t))X(t)^-,
\end{cases} \tag{12}$$

$$\begin{cases}
u_2^*(t,i,X(t)) = \hat{v}_{21}(P_1(t,i),\Lambda_1(t,i))X(t)^+ + \hat{v}_{22}(P_2(t,i),\Lambda_2(t,i))X(t)^-,\\
\beta_1^*(t,i,u_2(t),X(t)) = \hat{\beta}_{11}(P_1(t,i),\Lambda_1(t,i),v_2(t))X(t)^+\\
\qquad + \hat{\beta}_{12}(P_2(t,i),\Lambda_2(t,i),v_2(t))X(t)^-.
\end{cases} \tag{13}$$

Optimility

Theorem 8 (Solution for the constrained homogeneous game)

An optimal control-strategy pair for Player 1's (resp., Player 2's) value is given by(12) *(resp.,*(13)*). Moreover, the game has a value*

$$\widetilde{V}_1(x, i_0) = \widetilde{V}_2(x, i_0) = P_1(0, i_0)(x^+)^2 + P_2(0, i_0)(x^-)^2,$$

where $\big(P_k(\cdot, i), \Lambda_k(\cdot, i)\big)_{i\in\mathcal{M}}$, $k \in \{1, 2\}$, *are solutions of* (11).

Corollary 9

When $\widehat{R}_{12} = 0$, *the optimal strategies have nothing to do with the opponent's control:* $\beta_k^*(t, i, X(t)) = \hat{\beta}_{k1}(P_1(t, i), \Lambda_1(t, i))X(t)^+ + \hat{\beta}_{k2}(P_2(t, i), \Lambda_2(t, i))X(t)^-$.

Application to portfolio selection games

Problem formulation

- Two players' wealth processes:

$$\begin{cases} \mathrm{d}Y_k(t) = \Big[r(t,\alpha_t)Y_k(t) + \big[\mu_k(t,\alpha_t) - r(t,\alpha_t)\big]\pi_k(t)\Big]\,\mathrm{d}t \\ \qquad\qquad + \sigma_k(t,\alpha_t)\pi_k(t)\,\mathrm{d}W(t), \\ Y_k(0) = y_k,\ \alpha_0 = i_0 \in \mathcal{M}, \quad k \in \{1,2\}. \end{cases}$$

- Their difference $X(\cdot) \triangleq Y_1(\cdot) - Y_2(\cdot)$ satisfies

$$\begin{cases} \mathrm{d}X(t) = \Big[r(t,\alpha_t)X(t) + \big[\mu_1(t,\alpha_t) - r(t,\alpha_t)\big]\pi_1(t) \\ \qquad\qquad - \big[\mu_2(t,\alpha_t) - r(t,\alpha_t)\big]\pi_2(t)\Big]\,\mathrm{d}t \\ \qquad\qquad + \big[\sigma_1(t,\alpha_t)\pi_1(t) - \sigma_2(t,\alpha_t)\pi_2(t)\big]\,\mathrm{d}W(t), \\ X(0) = x \triangleq y_1 - y_2,\ \alpha_0 = i_0 \in \mathcal{M}. \end{cases} \tag{14}$$

Problem formulation

- Player 1 tries to narrow the gap $Y_1(T) - \frac{Y_1(T)+Y_2(T)}{2} = \frac{X(T)}{2}$.
- Player 2 tries to enlarge the gap.
- The functional of the zero-sum game is given as

$$J_{x,i_0}(\pi_1,\pi_2) = \mathbb{E}\Big[\int_0^T \Big(- R_1(t,\alpha_t)\pi_1(t)^2 + R_2(t,\alpha_t)\pi_2(t)^2\Big)\,\mathrm{d}t - \frac{1}{4}X(T)^2\Big],$$

where R_1, R_2 are uniformly positive.

Case 1: No trading constraint

- The SRE is

$$\begin{cases} \mathrm{d}P(t,i) = -\Big[2rP(t,i) - \dfrac{\Upsilon(P(t,i),\Lambda(t,i))}{\Theta(P(t,i),\Lambda(t,i))} + \sum_{j\in\mathcal{M}} q_{ij}P(t,j)\Big]\,\mathrm{d}t \\ \qquad\qquad + \Lambda(t,i)^\top\,\mathrm{d}W(t), \\ P(T,i) = -\dfrac{1}{4},\ |P(\cdot,i)| \leqslant \epsilon_2,\ \text{ for all }\ i\in\mathcal{M}, \end{cases}$$

where

$$\Phi_1(t,i,P,\Lambda) \triangleq P(\mu_1 - r) + \sigma_1\Lambda,\quad \Phi_2(t,i,P,\Lambda) \triangleq -P(\mu_2 - r) - \sigma_2\Lambda,$$

$$\Psi_1(t,i,P) \triangleq P\sigma_1\sigma_1^\top - R_1,\quad \Psi_2(t,i,P) \triangleq P\sigma_2\sigma_2^\top + R_2,\quad \Psi_3(t,i,P) \triangleq -P\sigma_1\sigma_2^\top,$$

$$\Theta(t,i,P) \triangleq \Psi_1\Psi_2 - \Psi_3^2 < 0,\quad \Upsilon(t,i,P,\Lambda) \triangleq \Psi_1\Phi_2^2 + \Psi_2\Phi_1^2 - 2\Psi_3\Phi_1\Phi_2.$$

Case 1: No trading constraint

The optimal control-strategy pairs are given as (π_1^*, β_2^*) for Player 1

$$\begin{cases} \pi_1^*(t,i,X(t)) = -\Upsilon_1(P(t,i),\Lambda(t,i))X(t)/\Theta(P(t,i)), \\ \beta_2^*(t,i,\pi_1(t),X(t)) = -\big[\Psi_3(P(t,i))\pi_1(t) + \Phi_2(P(t,i),\Lambda(t,i))X(t)\big]/\Psi_2(P(t,i)), \end{cases}$$

and (β_1^*, π_2^*) for Player 2

$$\begin{cases} \pi_2^*(t,i,X(t)) = -\Upsilon_2(P(t,i),\Lambda(t,i))X(t)/\Theta(P(t,i)), \\ \beta_1^*(t,i,\pi_2(t),X(t)) = -\big[\Psi_3(P(t,i))\pi_2(t) + \Phi_1(P(t,i),\Lambda(t,i))X(t)\big]/\Psi_1(P(t,i)). \end{cases}$$

Cases 2-4: Subject to no-shorting constraint

- The SREs are

$$\begin{cases} \mathrm{d}P_k(t,i) = -\Big[2rP_k(t,i) + \widetilde{G}_k(P_k(t,i),\Lambda_k(t,i)) + \sum_{j\in\mathcal{M}} q_{ij}P_k(t,j)\Big]\,\mathrm{d}t \\ \qquad\qquad\quad + \Lambda_k(t,i)^\top\,\mathrm{d}W(t), \\ P_k(T,i) = -\frac{1}{4},\ |P_k(\cdot,i)| \leqslant \epsilon_2, \ \ \text{for all } \ i\in\mathcal{M},\ k=\{1,2,3,4,5,6\}, \end{cases}$$

where

$$\widetilde{G}_1(t,i,P,\Lambda) \triangleq \frac{-(\Upsilon_1^+)^2 - \Theta\Phi_2^2}{\Theta\Psi_2},\ \ \widetilde{G}_2(t,i,P,\Lambda) \triangleq \frac{-(\Upsilon_1^-)^2 - \Theta\Phi_2^2}{\Theta\Psi_2},$$

$$\widetilde{G}_3(t,i,P,\Lambda) \triangleq \frac{-(\Upsilon_2^+)^2 - \Theta\Phi_1^2}{\Theta\Psi_1},\ \ \widetilde{G}_4(t,i,P,\Lambda) \triangleq \frac{-(\Upsilon_2^-)^2 - \Theta\Phi_1^2}{\Theta\Psi_1},$$

$$\widetilde{G}_5(t,i,P,\Lambda) \triangleq \big[(\Phi_1^+)^2 - 2\Phi_1\Phi_1^+\big]/\Psi_1 + \big[(\Phi_2^-)^2 + 2\Phi_2\Phi_2^-\big]/\Psi_2,$$

$$\widetilde{G}_6(t,i,P,\Lambda) \triangleq \big[(\Phi_1^-)^2 + 2\Phi_1\Phi_1^-\big]/\Psi_1 + \big[(\Phi_2^+)^2 - 2\Phi_2\Phi_2^+\big]/\Psi_2.$$

Case 2: Only Player 1 subject to no-shorting constraint

The optimal control-strategy pairs are given as (π_1^*, β_2^*) for Player 1

$$\begin{cases} \pi_1^*(t,i,X(t)) = -\Upsilon_1(P_1,\Lambda_1)^+ X(t)^+/\Theta(P_1) - \Upsilon_1(P_2,\Lambda_2)^- X(t)^-/\Theta(P_2), \\ \beta_2^*(t,i,\pi_1(t),X(t)) = -\big[\Psi_3(P_1)\pi_1(t)I_{\{X(t)>0\}} + \Phi_2(P_1,\Lambda_1)X(t)^+\big]/\Psi_2(P_1) \\ \qquad\qquad - \big[\Psi_3(P_2)\pi_1(t)I_{\{X(t)<0\}} - \Phi_2(P_2,\Lambda_2)X(t)^-\big]/\Psi_2(P_2), \end{cases}$$

and (β_1^*, π_2^*) for Player 2

$$\begin{cases} \pi_2^*(t,i,X(t)) = \dfrac{\big[\Psi_3(P_1)\Upsilon_1(P_1,\Lambda_1)^+ - \Phi_2(P_1,\Lambda_1)\Theta(P_1)\big]X(t)^+}{\Psi_2(P_1)\Theta(P_1)} \\ \qquad\qquad + \dfrac{\big[\Psi_3(P_2)\Upsilon_1(P_2,\Lambda_2)^- + \Phi_2(P_2,\Lambda_2)\Theta(P_2)\big]X(t)^-}{\Psi_2(P_2)\Theta(P_2)}, \\ \beta_1^*(t,i,\pi_2(t),X(t)) = -\big[\Psi_3(P_1)\pi_2(t)I_{\{X(t)>0\}} + \Phi_1(P_1,\Lambda_1)X(t)^+\big]^+/\Psi_1(P_1) \\ \qquad\qquad - \big[\Psi_3(P_2)\pi_2(t)I_{\{X(t)<0\}} - \Phi_1(P_2,\Lambda_2)X(t)^-\big]^+/\Psi_1(P_2). \end{cases}$$

Case 3: Only Player 2 subject to no-shorting constraint

The optimal control-strategy pairs are given as (π_1^*, β_2^*) for Player 1

$$\begin{cases} \pi_1^*(t,i,X(t)) = \dfrac{\big[\Psi_3(P_3)\Upsilon_2(P_3,\Lambda_3)^+ - \Phi_1(P_3,\Lambda_3)\Theta(P_3)\big]X(t)^+}{\Psi_1(P_3)\Theta(P_3)} \\ \qquad\qquad + \dfrac{\big[\Psi_3(P_4)\Upsilon_2(P_4,\Lambda_4)^- + \Phi_1(P_4,\Lambda_4)\Theta(P_4)\big]X(t)^-}{\Psi_1(P_4)\Theta(P_4)}, \\ \beta_2^*(t,i,\pi_1(t),X(t)) = \big[\Psi_3(P_3)\pi_1(t)I_{\{X(t)>0\}} + \Phi_2(P_3,\Lambda_3)X(t)^+\big]^-/\Psi_2(P_3) \\ \qquad\qquad + \big[\Psi_3(P_4)\pi_1(t)I_{\{X(t)<0\}} - \Phi_2(P_4,\Lambda_4)X(t)^-\big]^-/\Psi_2(P_4), \end{cases}$$

and (β_1^*, π_2^*) for Player 2

$$\begin{cases} \pi_2^*(t,i,X(t)) = -\Upsilon_2(P_3,\Lambda_3)^+X(t)^+/\Theta(P_3) - \Upsilon_2(P_4,\Lambda_4)^-X(t)^-/\Theta(P_4), \\ \beta_1^*(t,i,\pi_2(t),X(t)) = -\big[\Psi_3(P_3)\pi_2(t)I_{\{X(t)>0\}} + \Phi_1(P_3,\Lambda_3)X(t)^+\big]/\Psi_1(P_3) \\ \qquad\qquad - \big[\Psi_3(P_4)\pi_2(t)I_{\{X(t)<0\}} - \Phi_1(P_4,\Lambda_4)X(t)^-\big]/\Psi_1(P_4). \end{cases}$$

Case 4: Both subject to no-shorting constraint

The optimal control-strategy pairs are given as (π_1^*, β_2^*) for Player 1

$$\pi_1^*(t, i, X(t)) = \beta_1^*(t, i, X(t)) = \frac{\Phi_1(P_5, \Lambda_5)^+ X(t)^+}{-\Psi_1(P_5)} + \frac{\Phi_1(P_6, \Lambda_6)^- X(t)^-}{-\Psi_1(P_6)},$$

and (β_1^*, π_2^*) for Player 2

$$\pi_2^*(t, i, X(t)) = \beta_2^*(t, i, X(t)) = \frac{\Phi_2(P_5, \Lambda_5)^- X(t)^+}{\Psi_2(P_5)} + \frac{\Phi_2(P_6, \Lambda_6)^+ X(t)^-}{\Psi_2(P_6)}.$$

Thank you for your attention!

Q & A