Forward performance processes with unbounded market price of risk

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(joint work with Adrien Richou)

Celebrating Professor Ying Hu's 60th birthday, 19 June, 2024

Professor Ying Hu's influence on my academic journey

- Y. Hu, P. Imkeller and M. Muller (2005), Utility maximization in incomplete markets, *Ann. Appl. Probab.*, 15, pp. 1691-1712.
- P. Briand and Y. Hu (2006), BSDE with quadratic growth and unbounded terminal value, *Probab. Theory Related Fields*, 136, pp. 604-618.
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Professor Ying Hu's influence on my academic journey

Utility maximization in constrained and unbounded financial markets:

Applications to indifference valuation, regime switching, consumption and Epstein-Zin recursive utility

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Professor Ying Hu's influence on my academic journey

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Outline

The classical expected utility theory

- The key ingredients are the choices of the trading horizon $[0, T]$, and the investor's utility $u_T(x)$ at the maturity T.
- The objective is to maximize the expected utility of terminal wealth over the admissible strategies:

$$u(x, t; T) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_D[t,T]} \mathbb{E}[u_T(X_T^\pi) | \mathcal{F}_t, X_t = x]$$

- If $u_T(x)$ is power, log or exponential, then the value process $u(x, t; T)$ and its associated optimal strategy can be characterized in terms of the solution to quadratic BSDE defined on a finite horizon. See El Karoui and Rouge (2000, MF), Hu et al (2005, AOAP), Mania and Schweizer (2005, AOAP), Morlais (2009, FS), ... and textbooks by Pham (2009, Chapter 6.6) and Touzi (2013, Chapter 11.3).

- Once the investor's utility is chosen at T, her risk preferences cannot be revised. The problem is essentially a one-period problem even though the underlying portfolio choices are formulated in continuous time. See Musiela and Zariphopoulou (2003, preprint), (2008, MPRF), (2009, QF), (2010, SIFIN).
- The classical expected utility is also horizon biased: The investor will have an in-built preference for the time horizon.
  For example, suppose that power utility $u_T(x) = \frac{1}{\delta}x^\delta$, no trading constraints $\Pi = \mathbb{R}^d$, and constant market price of risk $\theta_t \equiv \theta$ for some constant θ. Then at any time τ, her value function is time monotone:

$$U(x, \tau; T) = \frac{x^\delta}{\delta} e^{-\frac{\delta}{2(1-\delta)}|\theta|^2(T-\tau)}.$$

  The investor will have an in-build preference over the time τ. In this case, she will choose $\tau = T$. See Henderson (2007, MAFE) and Henderson and Hobson (2007, SPA).

Forward performance processes with bounded market coefficients

- Forward performance process, dated back to 2003, was introduced by Musiela and Zariphopoulou in a series of their papers. See MZ (2008, MPRF), (2009, QF), (2010, SIFIN) $\cdots$

Definition

A forward performance process is an $\mathbb{F}$-progressively measurable process $U(x,t)$ such that

- for any $t \geq 0$, the map $x \mapsto U(x,t)$ is strictly increasing and strictly concave for $x \in \mathbb{D}$;
- Martingale optimality: for any admissible strategy $\pi \in \mathcal{A}_D$, and for any $0 \leq t \leq s < \infty$,

$$U(X_t^{\pi}, t) \geq \mathbb{E}\left[U(X_s^{\pi}, s)|\mathcal{F}_t\right],$$

and there exists an optimal strategy $\pi^* \in \mathcal{A}_D$ such that

$$U(X_t^{\pi^*}, t) = \mathbb{E}\left[U(X_s^{\pi^*}, s)|\mathcal{F}_t\right].$$

- From its definition, a forward performance process depends on market.
- Input: investor's initial risk preference+market; output: investor's forward performance process.
- In a market with $\mathbb{F}$ being the Brownian filtration and all the market coefficients are bounded and driven by a stochastic factor process, forward performance processes can be characterized in terms of the solutions of infinite horizon BSDE and ergodic BSDE. See Liang and Zariphopoulou (2017, SIFIN).
- In this talk, we consider a market with unbounded market coefficients, and characterize forward performance processes using infinite horizon quadratic BSDE and ergodic quadratic BSDE.

The stochastic factor model

- All the market coefficients depend on a d-dim stochastic factor process $V = (V^1, \cdots, V^d)^{tr}$:

$$dV_t^i = \eta^i(V_t)dt + \sum_{j=1}^{d} \kappa^{ij} dW_t^j.$$

- The drift $\eta(v)$ of the stochastic factor satisfies the dissipative condition:

$$(\eta(v) - \eta(\bar{v})) \cdot (v - \bar{v}) \leq -C_\eta |v - \bar{v}|^2$$

  for a large enough dissipative constant C_η (large dissipative condition).
- The volatility matrix κ of the stochastic factor is constant with $|\kappa| = 1$, and $\kappa\kappa^{tr}$ is positive definite.
- The market price of risk $\theta(v) = \sigma(v)^{tr}[\sigma(v)\sigma(v)^{tr}]^{-1}b(v)$ is uniformly bounded and Lipschitz continuous with Lipschitz constant C_θ.

Representation via SPDE

- Suppose that $U(x,t)$ admits the Itô's decomposition:

$$dU(x,t) = F(x,t)dt + Z(x,t) \cdot dW_t$$

  for some $\mathbb{F}$-progressively measurable processes $F(x,t)$ and $Z(x,t)$, and that all involved quantities have enough regularity.

- Applying Itô-Ventzell formula to $U(X_t(\pi), t)$ for any admissible π, we obtain that, for a chosen volatility $Z(x,t)$, the drift $F(x,t)$ must have a specific form:

$$F(x,t) = -\frac{1}{2}|x|^2 \partial_{xx} U(x,t) dist^2 \left\{ \Pi, -\frac{\partial_x Z(x,t) + \theta(V_t)\partial_x U(x,t)}{x \partial_{xx} U(x,t)} \right\} + \frac{|\partial_x Z(x,t) + \theta(V_t)\partial_x U(x,t)|^2}{2\partial_{xx} U(x,t)}.$$

- In general, solving the above fully nonlinear SPDE is a formidable task. See however El Karoui and Mrad (2014, SIFIN) for an equivalent formulation by using a duality and stochastic flow approach.

The case of zero volatility

- If $Z(x,t)=0$, the result is complete. See Musiela and Zariphopoulou (SIFIN, 2010). The SPDE reduces to

$$dU(x,t) = \frac{1}{2}|\theta(V_t)|^2 \frac{|U_x(x,t)|^2}{U_{xx}(x,t)}dt.$$

- Then $U(x,t) = u\left(x, \int_0^t |\theta(V)_s|^2 ds\right)$ where

$$u_t(x,t) = \frac{1}{2}\frac{|u_x(x,t)|^2}{u_{xx}(x,t)}.$$

- The solvability of the above PDE is closely related to the ill-posed heat equation:

$$h_t(x,t) + \frac{1}{2}h_{xx}(x,t) = 0.$$

The key ingredient is the Widder's theorem. See MZ (SIFIN, 2010).

- No boundedness assumption on $\theta(v)$.

Ergodic BSDE representation

Theorem (Liang and Zariphopoulou (2017, SIFIN))

Suppose $\theta(v)$ is bounded. The process $U(x,t) = \frac{x^\delta}{\delta} e^{f(V_t,t)} = \frac{x^\delta}{\delta} e^{Y_t - \lambda t}$ is a power forward performance process and satisfies

$$dU(x,t) = U(x,t)(-F(V_t, Z_t) + \frac{1}{2}|Z_t|^2)dt + U(x,t)Z_t \cdot dW_t.$$

The optimal strategy is given by $\pi_t^* = Proj_\Pi \frac{Z_t + \theta(V_t)}{(1-\delta)}$, where (Y, Z, λ) solves the ergodic BSDE with F:

$$dY_t = (-F(V_t, Z_t) + \lambda)dt + Z_t \cdot dW_t$$

with

$$F(V_t, z) = -\frac{1}{2}\delta(1-\delta) dist^2 \left\{ \Pi, \frac{z + \theta(V_t)}{1-\delta} \right\} + \frac{1}{2}\frac{\delta}{1-\delta}|z+\theta(V_t)|^2 + \frac{1}{2}|z|^2.$$

Moreover, the above equation admits a unique Markovian solution in the sense that $Y_t = y(V_t)$ for some function $y(\cdot)$ with linear growth, and $Z_t = z(V_t)$ with $|z(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$.

Connection with ergodic risk-sensitive optimisation

- The Markovian solution λ is the long-term growth rate of power utility maximisation problem:

$$\lambda = \sup_{\pi \in \mathcal{A}_D} \limsup_{T \to \infty} \frac{1}{T} \ln E^{\mathbb{P}^\pi} \left[\frac{X_T(\pi)^\delta}{\delta} \right].$$

  Moreover, the optimal control process is given as π^*.

- We need to first establish the ergodic risk-sensitive optimisation:

$$\lambda = \sup_{\pi \in \mathcal{A}_D} \limsup_{T \to \infty} \frac{1}{T} \ln E^{\mathbb{P}^\pi} \left(e^{\int_0^T L(V_u, \pi_u) du} \right),$$

  where $L(V_u, \pi_u) = -\frac{1}{2}\delta(1-\delta)|\pi_u|^2 + \delta\theta(V_u) \cdot \pi_u$, and the probability measure $\mathbb{P}^\pi$ is defined as

$$\frac{d\mathbb{P}^\pi}{d\mathbb{P}} = \mathcal{E}\left(\int_0^{\cdot} \delta\pi_u \cdot dW_u \right)_T.$$

Verification

- Apply Itô's formula to $U(X_t(\pi), t) = \frac{(X_t(\pi))^\delta}{\delta} e^{Y_t - \lambda t}$:

$$U(X_t(\pi), t) = U(X_0, 0) A_t^\pi \mathcal{E}\left(\int_0^\cdot (Z_u + \delta \pi_u) \cdot dW_u\right)_t$$

where

$$A_t^\pi = \exp\left(\int_0^t \left(-F(u, Z_u) - \frac{1}{2}\delta(1-\delta)|\pi_u - \frac{Z_u + \theta(V_u)}{1-\delta}|^2\right) du\right)$$
$$\times \exp\left(\int_0^t \left(\frac{1}{2}\frac{\delta}{1-\delta}|Z_u + \theta(V_u)|^2 + \frac{1}{2}|Z_u|^2\right) du\right).$$

- Therefore,

(1) with the choice of $F, A_t^\pi \leq 1$, $\downarrow$, and $A_t^{\pi^*} = 1$;

(2) $\mathcal{E}(\int_0^\cdot (Z_u + \delta\pi_u) \cdot dW_u)$ is in Class (D);

(3) $\pi_t^* = Proj_\Pi(\frac{Z_t + \theta(V_t)}{1-\delta}) \in \mathcal{A}_D$.

Ergodic BSDE

- The driver of the ergodic BSDE satisfies

$$
\begin{aligned}
|F(v,z)-F(\bar{v},z)| &\leq C_v(1+|z|)|v-\bar{v}|; \\
|F(v,z)-F(v,\bar{z})| &\leq C_z(1+|z|+|\bar{z}|)|z-\bar{z}|; \\
|F(v,0)| &\leq C_F.
\end{aligned}
$$

- To ensure Z is bounded, we require the dissipative constant $C_\eta > C_v$ (large dissipative condition).
- As a consequence, since $\pi_t^* = Proj_\Pi \frac{Z_t+\theta(V_t)}{(1-\delta)}$ and $\theta(\cdot)$ is bounded, if we require $\int_0^\cdot \pi_u \cdot dW_u$ is BMO, the verification will work.

Truncated ergodic BSDE

- Define a truncation function:

$$q(z) = \frac{\min\{|z|, C_v/(C_\eta - C_v)\}}{|z|} z \mathbf{1}_{\{z \neq 0\}}.$$

- Consider the truncated ergodic BSDE:

$$dY_t = (-F(V_t, q(Z_t)) + \lambda)dt + Z_t \cdot dW_t.$$

- The truncated driver is Lipschitz continuous:

$$|F(v, q(z)) - F(\bar{v}, q(z))| \leq \frac{C_\eta C_v}{C_\eta - C_v} |v - \bar{v}|;$$

$$|F(v, q(z)) - F(v, q(\bar{z}))| \leq C_z \frac{C_\eta + C_v}{C_\eta - C_v} |z - \bar{z}|.$$

- If we can prove the truncated ergodic BSDE with $F(\cdot, q(\cdot))$ admits a unique solution (Y, Z, λ) with $|Z_t| \leq \frac{C_v}{C_{\eta - C_v}}$, then $q(Z_t) = Z_t$, and therefore (Y, Z, λ) is also a solution to the original ergodic BSDE with $F(\cdot, \cdot)$.

Infinite horizon BSDE representation

Theorem (Liang and Zariphopoulou (2017, SIFIN))

Suppose that $\theta(v)$ is bounded. Then, the process $U^\rho(x,t) = \frac{x^\delta}{\delta} e^{f(V_t, \int_0^t V_s ds)} = \frac{x^\delta}{\delta} e^{Y_t^\rho - \int_0^t \rho Y_s^\rho ds}$ is a power forward performance process, and satisfies

$$dU^\rho(x,t) = U^\rho(x,t)(-F(V_t, Z_t^\rho) + \frac{1}{2}|Z_t^\rho|^2)dt + U^\rho(x,t)(Z_t^\rho) \cdot dW_t.$$

The optimal strategy is given by

$$\pi_t^* = Proj_\Pi \frac{Z_t^\rho + \theta(V_t)}{(1-\delta)},$$

where (Y^ρ, Z^ρ) solves the infinite horizon BSDE with F:

$$dY_t^\rho = (-F(V_t, Z_t^\rho) + \rho Y_t^\rho)dt + (Z_t^\rho) \cdot dW_t.$$

Moreover, the above equation admits a unique Markovian solution in the sense that $Y_t^\rho = y^\rho(V_t)$ for some function $y^\rho(\cdot)$ bounded by $\frac{K}{\rho}$, and $Z_t^\rho = z^\rho(V_t)$ with $|z^\rho(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$.

The case of unbounded market coefficients

Unbounded market price of risk

- The market price of risk $\theta(v)$ satisfies only Lipschitz continuity with Lipschitz constant C_θ.
- Example:
  - Underlying stock:

$$\frac{dS_t}{S_t} = b(V_t)dt + \sigma(V_t)dW_t^1,$$

    so the market price of risk $\theta(V_t) = b(V_t)/\sigma(V_t) = C_\theta V_t$.
  - Stochastic factor:

$$dV_t = -C_\eta V_t + \kappa^1 dW_t^1 + \kappa^2 dW_t^2.$$

  - The driver is

$$F(V_t, (z_1, z_2)^{tr}) = -C_\theta V_t z_1 - \frac{1}{2} C_\theta^2 |V_t|^2 + \frac{1}{2}|z_2|^2,$$

    where the trading constraint set $\Pi = \mathbb{R} \times \{0\}$ with $\pi_{1t} = \tilde{\pi}_t \sigma(V_t)$ and $\pi_{2t} \equiv 0$.

Ergodic quadratic BSDE

- Formally, it is expected that the ergodic BSDE representation still holds. However, the ergodic quadratic BSDE inherently exhibit quadratic growth.
- The driver of the ergodic quadratic BSDE satisfies

$$|F(v,z)-F(\bar{v},z)| \leq (C_v + k_v|z| + \frac{\alpha_v}{2}(|v|+|\bar{v}|))|v-\bar{v}|;$$
$$|F(v,z)-F(v,\bar{z})| \leq (C_z + k_z|v| + \frac{\alpha_z}{2}(|z|+|\bar{z}|))|z-\bar{z}|;$$
$$|F(v,0)| \leq C_F + \alpha_v|v|^2.$$

- When the market price of risk $\theta(v)$ is bounded, then $\alpha_v = k_z = 0$. It covers the ergodic BSDE studied in Liang and Zariphopoulou (2017, SIFIN).

Truncated ergodic quadratic BSDE

- Define a truncation function ρ_M: a smooth modification of the projection on the ball with radius M such that $|\rho_M| \leq M$, $|\nabla \rho_M| \leq 1$ and $\rho_M(x) = x$ for $|x| \leq M - 1$.
- The truncated driver satisfies

$$|F(\rho_M(v), z) - F(\rho_M(\bar{v}), z)| \leq (C_v + \alpha_v M + k_v |z|)|v - \bar{v}|;$$
$$|F(\rho_M(v), z) - F(\rho_M(v), \bar{z})| \leq (C_z + k_z M + \frac{\alpha_z}{2}(|z| + |\bar{z}|))|z - \bar{z}|;$$
$$|F(\rho_M(v), 0)| \leq C_F + \alpha_v M^2.$$

- The corresponding truncated ergodic quadratic BSDE with $F(\rho_M(\cdot, \cdot))$ admits a unique solution (Y, Z, λ) with Z bounded:

$$|z(\cdot)| \leq \frac{C_v + \alpha_v M}{C_\eta - C_v - \alpha_v M}$$

which only holds for $M \leq \frac{C_\eta - C_v}{\alpha_v}$ and will explode! So the previous results do not apply.

Enhanced Z estimations via De Giorgi type iteration

- The key is to analyze the following finite horizon discounted quadratic BSDE:

$$Y_t = \int_t^T [F(\rho_M(V_s), Z_s) - \rho Y_s]ds - \int_t^T Z_s \cdot dW_s,$$

  coupled with

$$dV_t = \eta(V_t)dt + \kappa \cdot dW_t.$$

- Aim to obtain estimate $|Z_t| \leq C(1 + |V_t|)$ where C is independent of ρ, M, T.
- Two steps:
  - $$|Z_t| \leq C_{M,T}$$

    for $C_{M,T}$ independent of ρ.
  - $$|Z_t| \leq C + \frac{\alpha_v}{C_\eta - k_v}|V_t|$$

    for C independent of ρ, M, T.

First step via variation

- The fist step is standard by variation

$$\nabla Y_t = \int_t^T [\nabla_v F(\rho_M(V_s), Z_s) \cdot \nabla V_s + \nabla_z F(\rho_M(V_s), Z_s) \cdot \nabla Z_s - \rho \nabla Y_s] ds - \int_t^T \nabla Z_s \cdot dW_s,$$

  and

$$\nabla V_t = 1 + \int_t^T \nabla \eta(V_s) \cdot \nabla V_s ds.$$

- Use the representation

$$Z_t = \nabla Y_t \cdot (\nabla V_t)^{-1} \cdot \kappa$$

  to obtain

$$|Z_t| \leq C_{M,T}.$$

Second step by De Giorgi type iteration

- Basic idea: Aim to show that

$$|Z_t| \leq A_n + B|V_t|,$$

with $A_0 = C_{M,T}$, and A_n is defined iteratively as

$$A_{n+1} = C + \beta A_n$$

for $\beta < 1$, and B, C, β, independent of ρ, M, T, are to be determined.

- Then, the sequence $\{A_n\}$ converges as a contraction function and $A_n \to C/(1-\beta)$, so that

$$|Z_t| \leq \frac{C}{1-\beta} + B|V_t|.$$

- In PDE literature, it is known as De Giorgi iteration technique, used to prove Hölder's regularity with measurable coefficients, maximum principle for weak solutions...
- In BSDE literature, this iteration technique was introduced by Richou (2012, SPA).

Second step by De Giorgi type iteration

- Change the probability measure in the variation equation

$$\nabla Y_t \cdot (\nabla V_t)^{-1} = E_t^{\mathbb{Q}^M}\left[\int_t^T e^{-\rho(s-t)} \nabla_v F(\rho_M(V_s), Z_s) \cdot \nabla V_s \cdot (\nabla V_t)^{-1} ds\right],$$

with $dW_s - \nabla_z F(\rho_M(V_s), Z_s)ds$ being BM under $\mathbb{Q}^M$.

- Suppose that $|Z_t| \leq A_n + B|V_t|$ holds for n, then

$$\begin{aligned} |Z_t| &= |\nabla Y_t \cdot (\nabla V_t)^{-1} \cdot \kappa| \\ &\leq E_t^{\mathbb{Q}^M}\left[\int_t^T e^{-\rho(s-t)}(C_v + \alpha_v|V_s| + k_v|Z_s|)e^{-C_\eta(s-t)} ds\right] \\ &\leq \frac{C_v + k_v A_n}{C_\eta} + (\alpha_v + k_v B)\int_t^T e^{-C_\eta(s-t)} E_t^{\mathbb{Q}^M}[|V_s|^2]^{\frac{1}{2}} ds \qquad (1) \\ &? \leq A_{n+1} + B|V_t|. \end{aligned}$$

- Hence, the question boils down to estimate $E_t^{\mathbb{Q}^M}[|V_s|^2]$, where

$$dV_t = [\eta(V_t) + \kappa \cdot \nabla_z F(\rho_M(V_t), Z_t)]dt + \kappa \cdot dW_t^{\mathbb{Q}^M}.$$

Second step by De Giorgi type iteration

- Define $\tilde{\epsilon} := C_\eta - k_z - \alpha_z B$ such that $\tilde{\epsilon} > 0$. Then, for any small $\epsilon < \frac{\tilde{\epsilon}}{2}$, suppose that $|Z_t| \leq A_n + B|V_t|$ holds for n,

$$E_t^{\mathbb{Q}^M}[|V_s|^2] \leq |V_t|^2 + \frac{\epsilon + |\eta(0)|^2 + C_z^2}{\epsilon(\tilde{\epsilon} - 2\epsilon)} + \frac{\alpha_z^2}{\tilde{\epsilon}(\tilde{\epsilon} - 2\epsilon)}|A_n|^2. \qquad (2)$$

Hence, (1)+(2) shows that $|Z_t|$ is bounded by

$$\underbrace{\frac{C_v}{C_\eta} + \frac{\alpha_v + k_v B}{C_\eta}\sqrt{\frac{\epsilon + |\eta(0)|^2 + C_z^2}{\epsilon(\tilde{\epsilon} - 2\epsilon)}}}_{=:C} + \underbrace{\left(\frac{k_v}{C_\eta} + \frac{\alpha_v + k_v B}{C_\eta}\frac{\alpha_z}{\sqrt{\tilde{\epsilon}(\tilde{\epsilon} - 2\epsilon)}}\right)}_{=:\beta<1} A_n + \underbrace{\frac{\alpha_v + k_v B}{C_\eta}}_{=:B}|V_t|$$

$$= (C + \beta A_n) + B|V_t|$$

$$= A_{n+1} + B|V_t|.$$

- Three constraints: $\tilde{\epsilon} > 0$, $\beta < 1$, and $\frac{\alpha_v + k_v B}{C_\eta} = B$.

Large dissipative condition

- Large dissipative condition:

$$C_\eta > k_v \vee k_z; \quad \frac{(C_\eta - k_v)^2 (C_\eta - k_z)}{2C_\eta - k_v} > \alpha_v \alpha_z.$$

- Then, $B = \frac{\alpha_v}{C_\eta - k_v}$, and

$$|Z_t| \le \frac{C}{1-\beta} + \frac{\alpha_v}{C_\eta - k_v}|V_t|.$$

- In the example, $k_v = k_z = 2C_\theta$, $\alpha_v = 2C_\theta^2$, $\alpha_z = 1$. So

$$C_\eta > 2C_\theta; \quad \frac{(C_\eta - 2C_\theta)^3}{C_\eta - C_\theta} > 4C_\theta^2.$$

 In particular, $C_\eta = pC_\theta$ for $p \ge 5$ satisfies the large dissipative condition. Asset is more stable but offers lower risk-adjusted return.

- When $\theta(v)$ is bounded, $\alpha_v = k_z = 0$, then $B = 0$, $C = C_v / C_\eta$, and $\beta = k_v / C_\eta$. Hence,

$$|Z_t| \le \frac{C}{1-\beta} = \frac{C_v}{C_\eta - k_v}$$

recovering the large dissipative condition in Liang and Zariphopoulou (2017, SIFIN).

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Happy Birthday Professor Ying Hu!