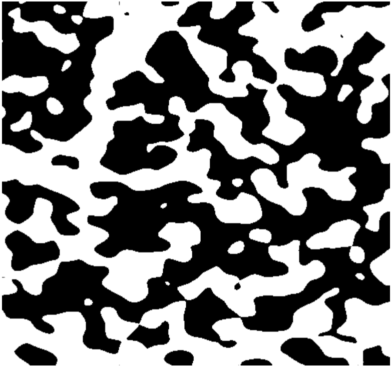
Box-crossing estimates for the nodal sets of planar Gaussian fields

Stephen Muirhead (Melbourne)

Rennes, June 2023



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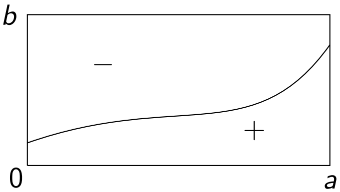

$$\{f=0\} \cap ([0,a]\times[0,b]) \text{ contains a path from } \{0\}\times[0,b] \text{ to } \{a\}\times[0,b]$$

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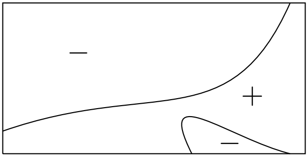

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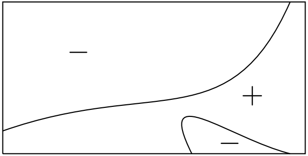
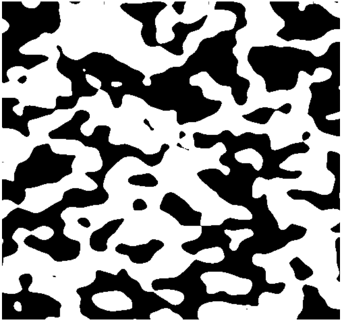
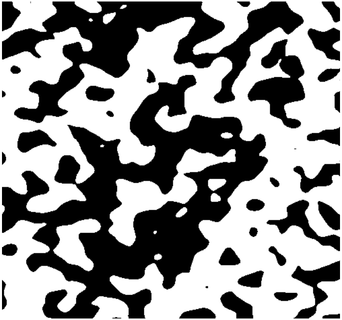


By continuity and non-degeneracy, almost surely

$$\text{NodalCross}(a, b) \implies \text{DomainCross}(a, b)$$

but not the converse.





Examples of DomainCross $\cap$ NodalCross^c.

We say that f satisfies the **nodal box-crossing (NBC) estimates** if, for every aspect ratio $\rho > 0$,

$$0 < \liminf_{R\to\infty} \mathrm{NodalCross}(R, \rho R) \le \limsup_{R\to\infty} \mathrm{NodalCross}(R, \rho R) < 1,$$

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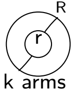
To show rigorously the existence of (sub-sequential) scaling limits one also needs **arm estimates** [Aizenman-Burchard '99]

$$\mathbb{P}[\mathrm{Arm}_k(r, R)] = \mathbb{P}\Big[$$



$$\Big] \leq c(r/R)^{2+\delta}$$

for some $\delta, k > 0$, which are so far unknown for any field.

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The KT result is much more general, applying to black/white colourings satisfying (i) **symmetry** under translation, axes reflection, and in black/white, and (ii) **positive associations**.

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$$K(x) = (1 + |x|^2)^{-\alpha/2} , \quad \alpha > 0.$$

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These results apply to more general fields, but [MV20] required some fairly strong assumptions (white noise decomposition).

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Rigorously, [KT '23] prove that

$$c_1 \leq \mathrm{DomainCross}(R, \rho R) \leq c_2$$

for $c_1(\rho), c_2(\rho) \in (0,1)$ **uniform** over all isotropic pos.-cor. fields.

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By contrast, correlations **weaken** NBC estimates, and we cannot expect uniform bounds (consider again the degenerate field).

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It applies to general fields with regularly varying covariance, under some assumptions (more later).

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Question. Is there a Harris-type criterion for NBC estimates that includes the Cauchy fields?

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$$\text{DBC estimates} + \text{`quasi-independence'} \implies \text{NBC estimates}$$

where QI means that, for disjoint domains D_1, D_2,

$$\lim_{R\to\infty} \sup_{\substack{A_1\in\sigma(RD_1), A_2\in\sigma(RD_2) \\ A_i \text{ crossing event}}} \Big| \mathbb{P}[A_1 \cap A_2] - \mathbb{P}[A_1]\mathbb{P}[A_2] \Big| = 0.$$

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One can also replace 'crossing event' with **monotone event**, which was exploited in [MV20] to lower to $\alpha > 2$.

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This generality comes at a cost: the **quantitative** bounds on the error are typically weaker than with the other approaches.

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Let X be a Gaussian vector in $\mathbb{R}^n$. Then for all $I_1, I_2 \subseteq \{1, \ldots, n\}$, increasing $A_i \in \sigma(I_1)$, and $\varepsilon > 0$,

$$\mathbb{P}[X \in A_1 \cap A_2] - \mathbb{P}[X + \varepsilon \in A_1]\mathbb{P}[X + \varepsilon \in A_2] \leq \frac{36\|K_{I_1,I_2}\|_\infty}{\varepsilon^2}.$$

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Question. Can the error be improved to $c_1 e^{-c_2\varepsilon^2/\|K_{I_1,I_2}\|_\infty}$?

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We combine with the following 'stability' estimate:

For increasing $A \in \sigma(I)$,

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Combining these, we obtain a (generic) mixing estimate

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This proves QI assuming $K \in L^1$, $\int K > 0$, and $K(R)R^2 \to 0$.

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Let D_1 and D_2 be unit balls separated by $T > 0$. Then

$$\sup_{\substack{A_1 \in \sigma(RD_1), A_2 \in \sigma(RD_2) \\ A_i \text{ monotone}}} \Big| \mathbb{P}[A_1 \cap A_2] - \mathbb{P}[A_1]\mathbb{P}[A_2] \Big| \le c_T$$

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for some explicit $c_T \to 0$ as $T \to \infty$.

This is sufficient to deduce NBC estimates for **sufficiently large** aspect ratio $\rho \gg 1$, but not all aspect ratios.

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We say f satisfies **weak ratio mixing** if, for every $\varepsilon > 0$ and disjoint domains D_1, D_2,

$$\liminf_{R\to\infty} \inf\Big\{\mathbb{P}[A_1\cap A_2] : A_i \in \sigma(RD_i) \text{ monotone}, \mathbb{P}[A_1] \geq \varepsilon, \mathbb{P}[A_2] \geq \varepsilon\Big\} > 0.$$

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Essentially we replace **asymptotic decorrelation** with **not asymptotic full correlation**.

We have

$$\text{DBC estimates} + \text{`weak ratio mixing'} \implies \text{NBC estimates.}$$

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Question. What is the true behaviour as $\alpha \to 0$? Does it decay?

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Then we apply the stability estimate

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To fix this we need to make two improvements:

1. Obtain a better decomposition with $\|g_R|_{RD_i}\|_\infty \leq cR^{-\alpha/2}$ (i.e. eliminate the '$\sqrt{\log}$' factor).
2. Apply some 'ratio' version of the stability estimate.

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To obtain a better decomposition we use a white noise representation that is **filtered-by-scale**.

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Fact. The Cauchy field has a scale-mixture decomposition with $w(t) = c_\alpha t^{-\alpha-3} e^{-1/(4t^2)}$ and $Q(x) = e^{-x^2}$, i.e. it is a scale mixture of Bargmann-Fock fields.

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Because the scale of the fluctuations of the contribution from $t \geq R$ is $\approx R$, we obtain

$$\|g_R|_{RD_i}\|_\infty \leq cR^{-\alpha/2} \quad \text{whp}$$

instead of the naive

$$\|g_R|_{RD_i}\|_\infty \leq cR^{-\alpha/2}\sqrt{\log R} \quad \text{whp.}$$

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To apply this we need to find a shift (h, h') in the CM space of (f_R, g_R) satisfying

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The shift we need is obtained from

$$\varphi(x,t) = \lambda(\mathbb{1}_{RD_1}(x) - \mathbb{1}_{RD_2}(x))\mathbb{1}_{t \in [aR, bR]}$$

for well chosen λ, a and b.

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Question. Find a more general criterion that doesn't require RV.

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Proposition (Sprinkled decoupling. M. '23)

Let X be a Gaussian vector in $\mathbb{R}^n$. Then for all $I_1, I_2 \subseteq \{1, \dots, n\}$, increasing $A_i \in \sigma(I_1)$, and $\varepsilon > 0$,

$$\mathbb{P}[X \in A_1 \cap A_2] - \mathbb{P}[X + \varepsilon \in A_1]\mathbb{P}[X + \varepsilon \in A_2] \leq \frac{36\|K_{I_1,I_2}\|_\infty}{\varepsilon^2}.$$

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We give the proof in the simpler case that $K_{I_1,I_2} \geq 0$.

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It has the key properties that

$$\frac{\partial T_A(X)}{\partial X_i} \geq 0 \qquad \text{and} \qquad \sum_i \frac{\partial T_A(X)}{\partial X_i} = 1.$$

Then by Gaussian interpolation

$$\mathrm{Cov}(T_{A_1}, T_{A_2}) = \int_0^\infty e^{-t} \sum_{1 \le i,j \le n} K(i,j) \mathbb{E}\Big[\frac{\partial T_{A_1}(X)}{\partial X_i} \frac{\partial T_{A_2}(X)}{\partial X_j}\Big] dt$$

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On the other hand, by Hoeffding's covariance formula

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On the other hand, by Hoeffding's covariance formula

$$\mathrm{Cov}(T_{A_1}, T_{A_2})$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{P}[T_{A_1} \le u, T_{A_2} \le v] - \mathbb{P}[T_{A_1} \le u]\mathbb{P}[T_{A_2} \le v]\, dudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{P}[X+u \in A_1, X+v \in A_2] - \mathbb{P}[X+u \in A_1]\mathbb{P}[X+v \in A_2]\, dudv$$

$$\ge \int_{0}^{\varepsilon}\int_{0}^{\varepsilon} \mathbb{P}[X+u \in A_1, X+v \in A_2] - \mathbb{P}[X+u \in A_1]\mathbb{P}[X+v \in A_2]\, dudv$$

where the last step used **positive associations**

$$\mathbb{P}[X + u \in A_1, X + v \in A_2] - \mathbb{P}[X + u \in A_1]\mathbb{P}[X + v \in A_2] \ge 0.$$

Putting this together

$$\int_0^{\varepsilon}\int_0^{\varepsilon} \mathbb{P}[X+u \in A_1, X+v \in A_2] - \mathbb{P}[X+u \in A_1]\mathbb{P}[X+v \in A_2]\, dudv$$

$$\leq \|K_{I_1,I_2}\|_\infty$$

Putting this together

$$\int_0^{\varepsilon}\int_0^{\varepsilon} \mathbb{P}[X+u \in A_1, X+v \in A_2] - \mathbb{P}[X+u \in A_1]\mathbb{P}[X+v \in A_2]\, dudv$$

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so (again by PA) there exists $u, v \in [0, \varepsilon]$ such that

$$\mathbb{P}[X+u \in A_1, X+v \in A_2] - \mathbb{P}[X+u \in A_1]\mathbb{P}[X+v \in A_2]$$

$$\leq \|K_{I_1,I_2}\|_\infty / \varepsilon^2.$$

Putting this together

$$\int_0^{\varepsilon}\int_0^{\varepsilon} \mathbb{P}[X+u \in A_1, X+v \in A_2] - \mathbb{P}[X+u \in A_1]\mathbb{P}[X+v \in A_2]\, du dv$$

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so (again by PA) there exists $u, v \in [0, \varepsilon]$ such that

$$\mathbb{P}[X+u \in A_1, X+v \in A_2] - \mathbb{P}[X+u \in A_1]\mathbb{P}[X+v \in A_2]$$

$$\leq \|K_{I_1,I_2}\|_\infty / \varepsilon^2.$$

By monotonicity the LHS is at least

$$\mathbb{P}[X \in A_1 \cap A_2] - \mathbb{P}[X+\varepsilon \in A_1]\mathbb{P}[X+\varepsilon \in A_2]$$

which ends the proof.

For the general result, the idea is to reduce to the case $K_{I_1,I_2} \geq 0$ by perturbing X with a small independent Gaussian vector Y.

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This works at the cost of increasing the constant from 1 to 36.

Question. Is the inequality true with constant 1 in general?

Thank you!

S. Muirhead, 'Percolation of strongly correlated Gaussian fields II: Sharpness of the phase transition', preprint, 2022

S. Muirhead, 'A sprinkled decoupling inequality for Gaussian vectors and applications', preprint, 2023