Uniform distribution for zeros of random polynomials

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Random Nodal Domains
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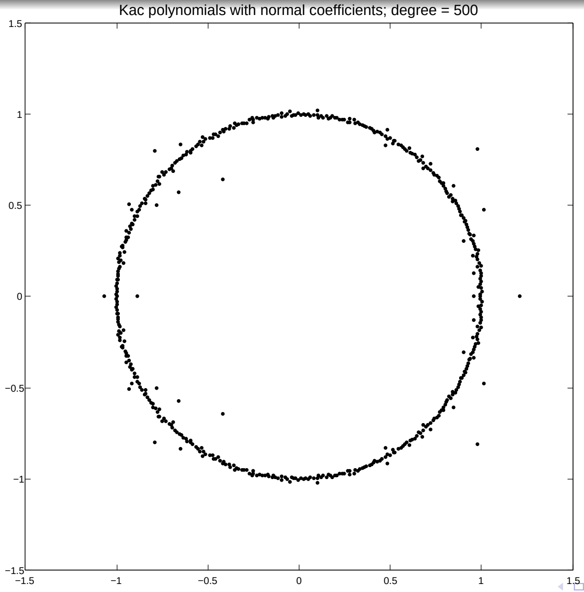
Gaussian coefficients, degree 500



Uniform zero distribution of random polynomials

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Let $\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{Z_k}$ and $d\mu_{\mathbb{T}}(e^{it}) := dt/(2\pi)$.

Question: When $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ with probability one (a.s.)?

Selected early results: Hammersley, 1956; Shparo&Shur, 1962; Arnold, 1966; Shepp&Vanderbei, 1995; Ibragimov&Zeitouni, 1997

C1 $A_k \in \mathbb{C}$ are i.i.d. r.v. with $\mathbb{P}(A_0 = 0) < 1$ and $\mathbb{E}[\log^+ |A_0|] < \infty$.

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C2 $A_k \in \mathbb{C}$ are identically distributed r.v. with $\mathbb{E}[|\log |A_0||] < \infty$.

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Random polynomials spanned by general bases

Let $E \subset \mathbb{C}$ be compact, $\text{cap}(E) > 0$, with the equilibrium measure μ_E. Define $||P_n||_E := \sup_{z\in E} |P_n(z)|$. Let $B_k(z) = \sum_{j=0}^{k} b_{j,k} z^j$, where $b_{j,k} \in \mathbb{C}$ and $b_{k,k} \neq 0$ for $k = 0, 1, 2, \ldots$. We assume that

$$\limsup_{k\to\infty} ||B_k||_E^{1/k} \le 1 \quad \text{and} \quad \lim_{k\to\infty} |b_{k,k}|^{1/k} = 1/\text{cap}(E).$$

These assumptions are satisfied for many orthonormal polynomials and other bases.

Pritsker, 2014-15: Suppose that E has empty interior and connected complement. If $\{A_k\}_{k=0}^{\infty}$ satisfy either **C1** or **C2**, then the zero counting measures for $P_n(z) = \sum_{k=0}^{n} A_k B_k(z)$ satisfy $\tau_n \xrightarrow{w} \mu_E$ a.s.

Dauvergne, 2021: For any compact set $E \subset \mathbb{C}$, $\text{cap}(E) > 0$, we have $\tau_n \xrightarrow{w} \mu_E$ a.s. $\Leftrightarrow$ **C1**, where **C1** means $A_k \in \mathbb{C}$ are i.i.d. r.v. with $\mathbb{P}(A_0 = 0) < 1$ and $\mathbb{E}[\log^+ |A_0|] < \infty$.

Required: For any bounded component G of $\mathbb{C} \setminus \partial E$, there is a point $w \in G$ such that

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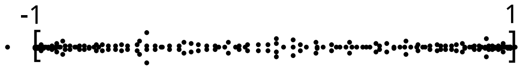
Example: Random orthogonal polynomials on $[a, b] \subset \mathbb{R}$

Let $E = [a, b] \subset \mathbb{R}$, and let $\{B_k\}_{k=0}^{\infty}$ be orthonormal with respect to a measure ν supported on $[a, b]$ such that $\nu' > 0$ a.e. on $[a, b]$. Consider the equilibrium measure of $[a, b]$:

$$d\mu_{[a.b]}(x) = \frac{dx}{\pi\sqrt{(x-a)(b-x)}}.$$

We have $\tau_n \xrightarrow{w} \mu_{[a,b]}$ a.s. for $P_n(z) = \sum_{k=0}^{n} A_k B_k(z)$ under either **C1** or **C2**.

Zeros of a random Legendre polynomial with $\mathcal{N}(0,1)$ coefficients:



Dependent coefficients

Let $B_k(z) = \sum_{j=0}^{k} b_{j,k} z^j$, where $b_{j,k} \in \mathbb{C}$ and $b_{k,k} \neq 0$ for $k = 0, 1, 2, \ldots$. Assume that

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Pritsker, 2015: Let $E \subset \mathbb{C}$ be compact, $\text{cap}(E) > 0$. If **C2** holds, and there is $t > 1$ s. t.

$$\sup_{z\in\mathbb{C}} \mathbb{E}\left[(\max(0, -\log|A_0 - z|))^t\right] < \infty, \qquad (1)$$

then the zero counting measures of $P_n(z) = \sum_{k=0}^{n} A_k B_k(z)$ satisfy $\tau_n \stackrel{w}{\to} \mu_E$ a.s.

Note: Condition (1) means that the probability measure of A_0 cannot be too concentrated at any point $z \in \mathbb{C}$, and it fails for, e.g., discrete random variables.

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Equidistribution of roots via balayage of measures

Let $G \in \mathbb{C}$ be a bounded open set, and let ν be a finite Borel measure supported on G. There is a unique *balayage* measure $\hat{\nu}$ supported on ∂G, of the same mass as ν, such that

$$\int \log|z-t|\,d\nu(t) = \int \log|z-t|\,d\hat{\nu}(t) \quad \text{for all } z \in \mathbb{C}\setminus\overline{G}.$$

If δ_w is the Dirac measure at $w \in G$, then $\widehat{\delta_w} = \omega(w, G)$ is the harmonic measure of G at w.

Given a compact set $E \in \mathbb{C}$, consider the unbounded component Ω of its complement $\mathbb{C}\setminus E$ and set $G := \mathbb{C}\setminus\overline{\Omega}$. For a zero counting measure τ_n, define its balayage out of G by

$$\widetilde{\tau}_n := \tau_n|_{\mathbb{C}\setminus G} + \widehat{\tau_n|_G} = \frac{1}{n}\left(\sum_{z_k\in\mathbb{C}\setminus G}\delta_{z_k} + \sum_{z_k\in G}\omega(z_k, G)\right),$$

where $\{z_k\}_{k=1}^n$ are the roots of of $P_n(z) = \sum_{k=0}^n A_k B_k(z)$.

Theorem: Let $E \subset \mathbb{C}$ be compact, $\text{cap}(E) > 0$, and keep the same assumptions on the deterministic basis. If $A_k \in \mathbb{C}$ are identically distributed r.v. with $\mathbb{E}[|\log|A_0||] < \infty$, then $\widetilde{\tau}_n \xrightarrow{w} \mu_E$ a.s.

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If δ_w is the Dirac measure at $w \in G$, then $\widehat{\delta_w} = \omega(w, G)$ is the harmonic measure of G at w.

Given a compact set $E \in \mathbb{C}$, consider the unbounded component Ω of its complement $\mathbb{C} \setminus E$ and set $G := \mathbb{C} \setminus \overline{\Omega}$. For a zero counting measure τ_n, define its balayage out of G by

$$\widetilde{\tau}_n := \tau_n|_{\mathbb{C}\setminus G} + \widehat{\tau_n|_G} = \frac{1}{n}\left(\sum_{z_k \in \mathbb{C}\setminus G} \delta_{z_k} + \sum_{z_k \in G} \omega(z_k, G)\right),$$

where $\{z_k\}_{k=1}^n$ are the roots of of $P_n(z) = \sum_{k=0}^n A_k B_k(z)$.

Theorem: Let $E \subset \mathbb{C}$ be compact, $\text{cap}(E) > 0$, and keep the same assumptions on the deterministic basis. If $A_k \in \mathbb{C}$ are identically distributed r.v. with $\mathbb{E}[|\log|A_0||] < \infty$, then $\widetilde{\tau}_n \xrightarrow{w} \mu_E$ a.s.

Equidistribution of roots via balayage of measures

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Expected discrepancy

Let $A_r(\alpha,\beta) = \{z : r < |z| < 1/r,\ \alpha \le \arg z < \beta\},\ 0 < r < 1.$

Pritsker, 2014: If $\{A_k\}_{k=0}^n$ satisfy $\mathbb{E}[|A_k|^t] \le c,\ k = 0,\dots,n$, for fixed $c, t > 0$, and $\mathbb{E}[\log |A_0|] > -\infty$, $\mathbb{E}[\log |A_n|] > -\infty$, then

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] \le C\sqrt{\frac{\log n}{n}}.$$

Equivalently, $\mathbb{E}[N_n(A_r(\alpha,\beta))] = \dfrac{\beta-\alpha}{2\pi} n + O\left(\sqrt{n \log n}\right).$

If $E \subset \mathbb{C}$ satisfies $\mathrm{dist}(E,\mathbb{T}) > 0$, then $\mathbb{E}[N_n(E)] = O(\log n)$.

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Zeros of lacunary random polynomials

Consider lacunary polynomials $L_n(z) = \sum_{k=0}^{n} A_k z^{r_k}$, where $\{r_k\}_{k=0}^{\infty} \subset \mathbb{N}$ are increasing and $\{A_n\}_{n=0}^{\infty} \subset \mathbb{C}$ are random variables.

Pritsker, 2018: Let $a > 0$ and $p \geq 1$. Suppose either $\{A_n\}_{n=0}^{\infty}$ are non-trivial i.i.d. random variables satisfying $\mathbb{E}[(\log^+ |A_n|)^{1/p}] < \infty$, or $\{A_n\}_{n=0}^{\infty}$ are identically distributed and $\mathbb{E}[|\log |A_n||^{1/p}] < \infty$.
If $r_n \sim an^p$ then $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.

Assume that $\liminf_{n\to\infty} r_n^{1/n} > 1$. If $\{A_n\}_{n=0}^{\infty} \subset \mathbb{C}$ are identically distributed and $\mathbb{E}[\log^+ |\log |A_n||] < \infty$, then $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.

Let $\{A_n\}_{n=0}^{\infty}$ be identically distributed with $\mathbb{E}[|A_n|^t] < \infty$ for a fixed $t \in (0, 1]$, and $\mathbb{E}[\log |A_n|] > -\infty$. If $\liminf_{n\to\infty} r_n^{1/n} = q > 1$ then

$$\limsup_{n\to\infty} \left| \tau_n(A_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi} \right|^{1/n} \leq \frac{1}{\sqrt{q}} \quad \text{a.s.}$$

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