Major/Minor MFG: Common Noise Helps!

Conference in honour of Ying Hu

June 18 2024

François Delarue

(Université Côte d'Azur, Nice, France)

Joint work with C. Mou (Hong-Kong)

European Research Council

Established by the European Commission

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1. Motivation

Major/Minor MFG

- Game between
  - major $\mathrm{d}X_t^0 = \alpha_t^0 \,\mathrm{d}t + \sigma_0 \,\mathrm{d}B_t^0$
  - many minors $\mathrm{d}X_t^i = \alpha_t^i \,\mathrm{d}t + \mathrm{d}B_t^i, \quad i = 1, \cdots, N$

Major/Minor MFG

- Game between
  - major $dX_t^0 = \alpha_t^0\, dt + \sigma_0\, dB_t^0$
  - many minors $dX_t^i = \alpha_t^i\, dt + dB_t^i, \quad i = 1, \cdots, N$
- Cost
  - major

$$\mathbb{E}^0\Big[g^0\Big(X_T^0, \frac{1}{N}\sum_{j=1}^N \delta_{X_T^j}\Big) + \int_0^T \Big\{f_t^0\Big(X_t^0, \frac{1}{N}\sum_{j=1}^N \delta_{X_T^j}\Big) + L^0(X_t^0, \alpha_t^0)\Big\}\, dt\Big]$$

  - minor i

$$\mathbb{E}^0\mathbb{E}\Big[g\Big(X_T^0, X_T^i, \frac{1}{N}\sum_{j=1}^N \delta_{X_T^j}\Big) + \int_0^T \Big\{f_t\Big(X_t^0, X_t^i, \frac{1}{N}\sum_{j=1}^N \delta_{X_T^j}\Big) + L(X_t^i, \alpha_t^i)\Big\}\, dt\Big]$$

Major/Minor MFG

- Game between
  - major $dX_t^0 = \alpha_t^0\, dt + \sigma_0\, dB_t^0$
  - many minors $dX_t^i = \alpha_t^i\, dt + dB_t^i, \quad i = 1, \cdots, N$
- Cost
  - major

$$\mathbb{E}^0\Big[g^0\Big(X_T^0, \frac{1}{N}\sum_{j=1}^{N}\delta_{X_T^j}\Big) + \int_0^T\Big\{f_t^0\Big(X_t^0, \frac{1}{N}\sum_{j=1}^{N}\delta_{X_T^j}\Big) + L^0(X_t^0, \alpha_t^0)\Big\}\, dt\Big]$$

  - minor i

$$\mathbb{E}^0\mathbb{E}\Big[g\Big(X_T^0, X_T^i, \frac{1}{N}\sum_{j=1}^{N}\delta_{X_T^j}\Big) + \int_0^T\Big\{f_t\Big(X_t^0, X_t^i, \frac{1}{N}\sum_{j=1}^{N}\delta_{X_T^j}\Big) + L(X_t^i, \alpha_t^i)\Big\}\, dt\Big]$$

- MFG is to directly address the game obtained by letting $N \to \infty$ formally and just using the statistical description of the minors

Bibliography

• Introduced by Huang (2009)

• Variants by Bensoussan et al. (2015, 16), Buckdahn et al. (2014), Caines et al. (2014, 20)

• Master Equation and Markov Nash Equilibria: Lasry & Lions (2018), Cardaliaguet & Cirant & Porretta (2020, 23)

  ◦ controls in Markov feedback form

  ◦ PDE characterisation for the value of the game, using Master Equations (PDEs on the space of probability measures)

• Open loop vs. Markov feedback form: Carmona & Zhu (2014), Carmona & Wang (2017)

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2. Characteristics

Stochastic forward-backward system

- Verification argument starting from (V^0, V): values of the game
  - $V^0(t, x_0, \mu)$: equilibrium cost to major when initial position at time t is x_0 and initial state of minors is μ (probability measure)
  - $V(t, x_0, x, \mu)$: equilibrium cost to one tagged minor starting from x at time t

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  ◦ $V(t, x_0, x, \mu)$: equilibrium cost to one tagged minor starting from x at time t

- If (V^0, V) smooth

$$
\begin{aligned}
dX_t^0 &= -\nabla_p H^0(X_t^0, \nabla_{x_0} V^0(t, X_t^0, \mu_t))\,dt + \sigma_0\,dB_t^0 \\
\partial_t \mu_t - \tfrac{1}{2}\Delta_x \mu_t - \mathrm{div}_x\big(\mu_t \nabla_p H(x, \nabla_x V(t, X_t^0, \cdot, \mu_t))\big) &= 0
\end{aligned}
$$

  ◦ Hamiltonians

$$
\begin{aligned}
H^0(x_0, p) &:= \sup_{\alpha \in \mathbb{R}^d} \big[-p \cdot \alpha - L^0(x, \alpha)\big] \\
H(x, p) &= \sup_{\alpha \in \mathbb{R}^d} \big[-p \cdot \alpha - L(x, \alpha)\big]
\end{aligned}
$$

  ◦ statistical state of the minors become random under noise of the major!

Stochastic forward-backward system

- If (V^0, V) smooth

$$dX_t^0 = -\nabla_p H^0(X_t^0, \nabla_{x_0} V^0(t, X_t^0, \mu_t))\, dt + \sigma_0\, dB_t^0$$

$$\partial_t \mu_t - \frac{1}{2}\Delta_x \mu_t - \mathrm{div}_x\big(\mu_t \nabla_p H(x, \nabla_x V(t, X_t^0, \cdot, \mu_t))\big) = 0$$

- Equation for the major
  - $Y_t^0 := V^0(t, X_t^0, \mu_t), \quad Z_t^0 := \nabla_x V^0(t, X_t^0, \mu_t)$

$$dX_t^0 = -\nabla_p H^0(X_t^0, Z_t^0)\, dt + \sigma_0\, dB_t^0$$

$$dY_t^0 = -\Big(f_t^0(X_t^0, \mu_t) + L^0(X_t^0, -\nabla_p H^0(X_t^0, Z_t^0))\Big)\, dt + \sigma_0 Z_t^0 \cdot dB_t^0$$

$$X_0^0 = x_0, \quad Y_T^0 = g^0(X_T^0, \mu_T)$$

Stochastic forward-backward system

- Equation for the major
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\end{aligned}
$$

- Equation for the minor
  - $u_t(x) := V(t, X_t^0, x, \mu_t), \quad v_t^0(x) := \nabla_{x_0} V(t, X_t^0, x, \mu_t)$

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$$\mathrm{d}_t u_t(x) = \Big(-\tfrac{1}{2}\Delta_x u_t(x) + H(x, \nabla_x u_t(x)) - f_t(X_t^0, x, \mu_t)\Big)\,\mathrm{d}t + \sigma_0 v_t^0(x) \cdot \mathrm{d}B_t^0$$

$$\mu_0 = \mu, \quad u_T(x) = g(X_T^0, x, \mu_T)$$

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$$X_0^0 = x_0, \quad Y_T^0 = g^0(X_T^0, \mu_T)$$

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  - $u_t(x) := V(t, X_t^0, x, \mu_t), \quad v_t^0(x) := \nabla_{x_0} V(t, X_t^0, x, \mu_t)$

$$\partial_t \mu_t - \frac{1}{2}\Delta_x \mu_t - \mathrm{div}_x\Big(\nabla_p H(\cdot, \nabla_x u_t)\mu_t\Big) = 0$$
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$$\mu_0 = \mu, \quad u_T(x) = g(X_T^0, x, \mu_T)$$

3. Main Results

Program

- Connection between MFG and Characteristics

    ◦ ∃! to Stochastic FB system for any initial condition $(t, x_0, \mu) \Rightarrow$ ∃! Nash equilibrium

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- Example: $T \ll 1$

  ∘ prove ∃! to Stochastic FB under Cauchy-Lipschitz type conditions for

$$T \leq c := c\Big(\mathrm{Lip}_{x_0,x,\mu}(g^0, g), \text{...others...}\Big)$$

  ∘ makes it possible to define (V^0, V)

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- On arbitrary length T: a priori estimates for

$$\mathrm{Lip}_{x_0,x,\mu}(V^0(t,\cdot,\cdot), V(t,\cdot,\cdot,\cdot)) \qquad \text{under} \quad \sigma_0^2 \gg 1$$

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- Combine with smoothness to get classical solution to master equations

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- Combine with smoothness to get classical solution to master equations

Assumptions

- Major in $\mathbb{R}^d$, minor in $\mathbb{T}^d$
- Standard assumptions
  - coefficients are $C^{d/2+2+\cdots}$, bounded derivatives in x_0, x
  - (strictly) convex Hamiltonians, ∇_p at most of linear growth in p, other derivatives are bounded
- Monotonicity f, g

$$\langle f(x_0,\cdot,\mu') - f(x_0,\cdot,\mu), \mu' - \mu\rangle \geq 0$$

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$$\langle f(x_0, \cdot, \mu') - f(x_0, \cdot, \mu), \mu' - \mu \rangle \geq 0$$

- Then, existence and uniqueness if $\sigma_0^2 \geq C(T)$
- To get $C(\not{T})$
  - long time behaviour of f (to get $\sigma_0^2 \gg 1$ indep of T)

$$\int_0^\infty \sup_{x_0 \in \mathbb{T}^d} \sup_{\mu \in \mathcal{P}(\mathbb{T}^d)} |f_t^0(x_0, \mu) - F^0(x_0)| \, \mathrm{d}t < \infty$$

$$\int_0^\infty \sup_{x, x_0 \in \mathbb{T}^d} \sup_{\mu \in \mathcal{P}(\mathbb{T}^d)} |f_t(x_0, x, \mu) - F(x, \mu)| \, \mathrm{d}t < \infty$$

4. A priori Estimates

Easier Setting

- Choose
  - $f^0 \equiv 0, f \equiv 0$
  - $L^0(x_0, \alpha_0) = \frac{1}{2}|\alpha_0|^2$, $L(x, \alpha) = \frac{1}{2}|\alpha|^2$

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- Stochastic FB system becomes
  - major

$$
\begin{aligned}
\mathrm{d}X_t^0 &= -Z_t^0 \,\mathrm{d}t + \sigma_0 \,\mathrm{d}B_t^0 \\
\mathrm{d}Y_t^0 &= -\frac{1}{2}|Z_t^0|^2 \,\mathrm{d}t + \sigma_0 Z_t^0 \cdot \mathrm{d}B_t^0 \\
Y_T^0 &= g^0(X_T^0, \mu_T)
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$$\begin{aligned} dX_t^0 &= -Z_t^0 \, dt + \sigma_0 \, dB_t^0 \\ dY_t^0 &= -\tfrac{1}{2}|Z_t^0|^2 \, dt + \sigma_0 Z_t^0 \cdot dB_t^0 \\ Y_T^0 &= g^0(X_T^0, \mu_T) \end{aligned}$$

  - minor

$$\begin{aligned} &\partial_t \mu_t - \tfrac{1}{2}\Delta_x \mu_t - \operatorname{div}_x\Big(\nabla_x u_t \mu_t\Big) = 0 \\ &d_t u_t(x) = \Big(-\tfrac{1}{2}\Delta_x u_t(x) + \tfrac{1}{2}|\nabla_x u_t(x)|^2\Big) \, dt + \sigma_0 v_t^0(x) \cdot dB_t^0 \\ &u_T(x) = g(X_T^0, x, \mu_T) \end{aligned}$$

A priori Bounds I

- Bounds on u
  - take ∂_x in backward(u) and use maximum principle

$$\sup_{t\in[0,T]} |\nabla_x u_t| \leq C(\sigma_0, T)$$

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$$\sup_{t\in[0,T]} |\nabla_x u_t| \leq C(\sigma_0, T)$$

  - take higher derivatives

$$\sup_{t\in[0,T]} \|\nabla_x u_t\|_{d/2+1+\cdots} \leq C(\sigma_0, T)$$

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- Bounds on u
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$$\sup_{t\in[0,T]} \|\nabla_x u_t\|_{d/2+1+\cdots} \leq C(\cancel{\sigma_0}, \cancel{T})$$

- Bounds on v^0
  - for any stopping time τ

$$\sigma_0^2 \mathbb{E}^0\Big[\int_\tau^T \Big\| v^0(r,\cdot) - \int_{\mathbb{T}^d} v^0(r,x)dx \Big\|_1^2 \mathrm{d}r \,|\, \mathcal{F}_\tau^0\Big] \leq C(\cancel{\sigma_0}, \cancel{T})$$

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  - BMO estimates

$$\mathbb{E}^0\Big[\exp\Big(\gamma\sigma_0^2 \int_\tau^T \Big\| v^0(r,\cdot) - \int_{\mathbb{T}^d} v^0(r,x)\ \mathrm{d}x \Big\|_1^2\ \mathrm{d}r\Big) \,|\, \mathcal{F}_\tau^0\Big] \leq C(\cancel{\sigma_0}, \cancel{T})$$

for some $\gamma = \gamma(\cancel{\sigma_0}, \cancel{T}) > 0$

A priori Bounds II

- Bounds on Z^0
  - for any stopping time τ

$$\mathbb{E}^0\Big[\int_\tau^T \sigma_0^2 |Z_r^0|^2 \,\mathrm{d}r \,\big|\, \mathcal{F}_\tau^0\Big] \leq C(\cancel{\sigma_0}, \cancel{T})$$

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$$\mathbb{E}^0\Big[\exp\Big(\gamma_0 \sigma_0^2 \int_\tau^T |Z_r^0|^2 \, \mathrm{d}r\Big) \,|\, \mathcal{F}_\tau^0\Big] \leq C_0(\cancel{\sigma_0}, \cancel{T})$$

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for some $\gamma_0 = \gamma_0(\cancel{\sigma_0}, \cancel{T}) > 0$

- Application
  - density on $(\Omega^0, \mathbb{P}^0)$

$$\mathcal{E}_t^0 := \exp\Big(\int_0^t \sigma_0^{-1} Z_s^0 \cdot \mathrm{d}B_s^0 - \frac{1}{2}\int_0^t \sigma_0^{-2}|Z_s^0|^2 \, \mathrm{d}s\Big), \quad t \in [0, T],$$

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$$\mathcal{E}_t^0 := \exp\Big(\int_0^t \sigma_0^{-1} Z_s^0 \cdot \mathrm{d}B_s^0 - \frac{1}{2}\int_0^t \sigma_0^{-2}|Z_s^0|^2 \, \mathrm{d}s\Big), \quad t \in [0,T],$$

  - bound

$$\mathbb{E}\Big[(\mathcal{E}_T^0(\mathcal{E}_t^0)^{-1})^{1+\gamma_0} \,|\, \mathcal{F}_t^0\Big] \leq C_0$$

5. Weak Formulation

Girsanov transformation

- New probability measure $\tilde{\mathbb{P}}^0 := \mathcal{E}_T \cdot \mathbb{P}^0$
  - tilted Brownian motion

$$\widetilde{B}^0_t := B^0_t - \sigma_0^{-1} \int_0^t Z^0_s \, \mathrm{d}s$$

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- New Stochastic FB system
  - major

$$\begin{aligned} \mathrm{d}X^0_t &= \sigma_0 \, \mathrm{d}\widetilde{B}^0_t \\ \mathrm{d}Y^0_t &= \tfrac{1}{2}|Z^0_t|^2 \, \mathrm{d}t + \sigma_0 Z^0_t \cdot \mathrm{d}\widetilde{B}^0_t \\ Y^0_T &= g^0(X^0_T, \mu_T) \end{aligned}$$

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  - minor

$$\partial_t \mu_t - \tfrac{1}{2}\Delta_x \mu_t - \mathrm{div}_x\Big(\nabla_x u_t \mu_t\Big) = 0$$

$$\mathrm{d}_t u_t(x) = \Big(-\tfrac{1}{2}\Delta_x u_t(x) + \tfrac{1}{2}|\nabla_x u_t(x)|^2 + Z_t^0 \cdot v_t^0(x)\Big)\,\mathrm{d}t + \sigma_0 v_t^0(x) \cdot \mathrm{d}\widetilde{B}_t^0$$

$$u_T(x) = g(X_T^0, x, \mu_T)$$

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Y_T^0 &= g^0(X_T^0, \mu_T)
\end{aligned}
$$

  - minor

$$
\begin{aligned}
&\partial_t \mu_t - \tfrac{1}{2}\Delta_x \mu_t - \mathrm{div}_x\Big(\nabla_x u_t \mu_t\Big) = 0 \\
&\mathrm{d}_t u_t(x) = \Big(-\tfrac{1}{2}\Delta_x u_t(x) + \tfrac{1}{2}|\nabla_x u_t(x)|^2 + Z_t^0 \cdot v_t^0(x)\Big)\,\mathrm{d}t + \sigma_0 v_t^0(x) \cdot \mathrm{d}\widetilde{B}_t^0 \\
&u_T(x) = g(X_T^0, x, \mu_T)
\end{aligned}
$$

- Still have a priori bounds!

$$
\tilde{\mathbb{E}}^0\left[\exp\left(\tilde{\gamma}\sigma_0^2 \int_\tau^T \left\| v^0(r,\cdot) - \int_{\mathbb{T}^d} v^0(r,x)\,\mathrm{d}x \right\|_1^2 \,\mathrm{d}r\right) \Big| \mathcal{F}_\tau^0\right] \leq C(\sigma_0, T)
$$

Flow in the weak formulation

- Take two initial conditions (x_0, μ_0) and (x_0', μ_0')

  ◦ goal is to address

$$Y_0 - Y_0' \quad \text{and} \quad u_0 - u_0'$$

  ◦ same

$$V_0(0, x_0, \mu_0) - V_0(0, x_0', \mu_0') \quad \text{and} \quad V(0, x_0, \cdot, \mu_0) - V(0, x_0', \cdot, \mu_0')$$

  ◦ Markov structure ⇒ licit to make this on the weak formulation with the same Brownian motion for all the initial conditions

Flow in the weak formulation

- Take two initial conditions (x_0, μ_0) and (x_0', μ_0')
  - goal is to address

$$Y_0 - Y_0' \quad \text{and} \quad u_0 - u_0'$$

- Write $\delta X_t^0 = X_t^{0,\prime} - X_t^0, \delta Y_t^0 = Y_t^{0,\prime} - Y_t^0 \ldots$

Flow in the weak formulation

- Take two initial conditions (x_0, μ_0) and (x'_0, μ'_0)
- Write $\delta X^0_t = X^{0,\prime}_t - X^0_t$, $\delta Y^0_t = Y^{0,\prime}_t - Y^0_t$ ...
  - difference of the majors

$$
\begin{aligned}
d\delta X^0_t &= 0, \\
d\delta Y^0_t &= \frac{Z^0_t + Z^{0,\prime}_t}{2} \cdot \delta Z^0_t \, dt + \sigma_0 \delta Z^0_t \cdot dB^0_t
\end{aligned}
$$

Flow in the weak formulation

- Take two initial conditions (x_0, μ_0) and (x_0', μ_0')
- Write $\delta X_t^0 = X_t^{0,\prime} - X_t^0$, $\delta Y_t^0 = Y_t^{0,\prime} - Y_t^0$ ...
  - difference of the majors

$$
\begin{aligned}
d\delta X_t^0 &= 0, \\
d\delta Y_t^0 &= \frac{Z_t^0 + Z_t^{0,\prime}}{2} \cdot \delta Z_t^0 \, dt + \sigma_0 \delta Z_t^0 \cdot dB_t^0
\end{aligned}
$$

  - difference of the minors

$$
\partial_t \delta\mu_t - \tfrac{1}{2}\Delta_x \delta\mu_t - \mathrm{div}_x\Big(\nabla_x \frac{u_t + u_t'}{2}\delta\mu_t\Big) - \mathrm{div}_x\Big(\nabla_x \delta u_t \frac{\mu_t + \mu_t'}{2}\Big) = 0
$$

$$
\begin{aligned}
d_t \delta u_t(x) = \Big(&-\tfrac{1}{2}\Delta_x \delta u_t(x) + \nabla_x \frac{u_t(x) + u_t'(x)}{2} \cdot \nabla_x \delta u_t(x) \\
&+ \frac{Z_t^0 + Z_t^{0,\prime}}{2} \cdot \delta v_t^0(x) + \delta Z_t^0 \cdot \frac{v_t^0(x) + v_t^{0,\prime}(x)}{2}\Big) dt + \sigma_0 \delta v_t^0(x) \cdot dB_t^0
\end{aligned}
$$

Flow in the weak formulation

- Take two initial conditions (x_0, μ_0) and (x'_0, μ'_0)
- Write $\delta X_t^0 = X_t^{0,\prime} - X_t^0$, $\delta Y_t^0 = Y_t^{0,\prime} - Y_t^0 \dots$
  - difference of the majors

$$\mathrm{d}\delta X_t^0 = 0,$$
$$\mathrm{d}\delta Y_t^0 = \frac{Z_t^0 + Z_t^{0,\prime}}{2} \cdot \delta Z_t^0 \,\mathrm{d}t + \sigma_0 \delta Z_t^0 \cdot \mathrm{d}B_t^0$$

  - difference of the minors

$$\partial_t \delta\mu_t - \tfrac{1}{2}\Delta_x \delta\mu_t - \mathrm{div}_x\Big(\nabla_x \frac{u_t + u'_t}{2}\delta\mu_t\Big) - \mathrm{div}_x\Big(\nabla_x \delta u_t \frac{\mu_t + \mu'_t}{2}\Big) = 0$$
$$\mathrm{d}_t \delta u_t(x) = \Big(-\tfrac{1}{2}\Delta_x \delta u_t(x) + \nabla_x \frac{u_t(x) + u'_t(x)}{2} \cdot \nabla_x \delta u_t(x) + \frac{Z_t^0 + Z_t^{0,\prime}}{2} \cdot \delta v_t^0(x) + \delta Z_t^0 \cdot \frac{v_t^0(x) + v_t^{0,\prime}(x)}{2}\Big)\mathrm{d}t + \sigma_0 \delta v_t^0(x) \cdot \mathrm{d}B_t^0$$

6. Linearized system

Tilted linearized system

- Under new probability measure $\bar{\mathbb{P}}^0$
  - difference of the majors

$$
\begin{aligned}
\mathrm{d}\delta X_t^0 &= 0\\
\mathrm{d}\delta Y_t^0 &= \sigma_0 \delta Z_t^0 \cdot \mathrm{d}\bar{B}_t^0\\
\delta Y_T^0 &= g^0(X_T^0, \mu_T) - g^0(X_T^{0,\prime}, \mu_T')
\end{aligned}
$$

  - difference of the minors

$$
\partial_t \delta\mu_t - \tfrac{1}{2}\Delta_x \delta\mu_t - \mathrm{div}_x\Big(\nabla_x \frac{u_t + u_t'}{2}\delta\mu_t\Big) - \mathrm{div}_x\Big(\nabla_x \delta u_t \frac{\mu_t + \mu_t'}{2}\Big) = 0
$$

$$
\begin{aligned}
\mathrm{d}_t \delta u_t(x) = \Big(&-\tfrac{1}{2}\Delta_x \delta u_t(x) + \nabla_x \frac{u_t(x) + u_t'(x)}{2} \cdot \nabla_x \delta u_t(x)\\
&+ \delta Z_t^0 \cdot \frac{v_t^0(x) + v_t^{0,\prime}(x)}{2}\Big)\mathrm{d}t + \sigma_0 \delta v_t^0(x) \cdot \mathrm{d}\bar{B}_t^0
\end{aligned}
$$

$$
\delta u_T(x) = g(X_T^{0,\prime}, x, \mu_T') - g(X_T^0, x, \mu_T)
$$

Key functional - I

- Follow Lasry-Lions proof of stability for standard MFGs

$$
\begin{aligned}
&\mathrm{d}_t[\varepsilon_1|\delta Y_t|^2 - (\delta u_t, \delta\mu_t)] \\
&= \Big[\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x \delta u_t|^2, \frac{\mu_t + \mu_t'}{2}) - \Big(\delta\mu_t, \delta Z_t^0 \cdot \frac{v_t^0 + v_t^{0,\prime}}{2}\Big)\Big] \\
&\qquad + \mathrm{dmart}_t
\end{aligned}
$$

  - intuitively, needs the dt term in RHS to be ≥ 0

Key functional - I

- Follow Lasry-Lions proof of stability for standard MFGs

$$\mathrm{d}_t[\varepsilon_1|\delta Y_t|^2 - (\delta u_t, \delta\mu_t)]$$
$$= \Big[\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x \delta u_t|^2, \frac{\mu_t + \mu_t'}{2}) - \Big(\delta\mu_t, \delta Z_t^0 \cdot \frac{v_t^0 + v_t^{0,\prime}}{2}\Big)\Big]$$
$$+ \mathrm{dmart}_t$$

  - intuitively, needs the $\mathrm{d}t$ term in RHS to be ≥ 0

- Need penalization to kill bad term

$$e_t = \overline{\mathbb{E}}^0\Big[\exp\Big(-\frac{\sigma_0^2}{A}\int_t^T \Big\|\frac{v_r^0 + v_r^{0,\prime}}{2} - \int_{\mathbb{T}^d}\frac{v_r^0 + v_r^{0,\prime}}{2}\Big\|_1^2 \,\mathrm{d}r\Big) \,|\, \mathcal{F}_t^0\Big], \quad t \in [0, T]$$

Key functional - I

- Follow Lasry-Lions proof of stability for standard MFGs

$$
\begin{aligned}
&\mathrm{d}_t\big[\varepsilon_1 |\delta Y_t|^2 - (\delta u_t, \delta\mu_t)\big] \\
&= \Big[\varepsilon_1 \sigma_0^2 |\delta Z_t^0|^2 + \big(|\nabla_x \delta u_t|^2, \frac{\mu_t + \mu_t'}{2}\big) - \Big(\delta\mu_t, \delta Z_t^0 \cdot \frac{v_t^0 + v_t^{0,\prime}}{2}\Big)\Big] \\
&\qquad + \mathrm{dmart}_t
\end{aligned}
$$

  - intuitively, needs the dt term in RHS to be ≥ 0

- Need penalization to kill bad term

$$
e_t = \overline{\mathbb{E}}^0\Big[\exp\Big(-\frac{\sigma_0^2}{A}\int_t^T \Big\| \frac{v_r^0 + v_r^{0,\prime}}{2} - \fint_{\mathbb{T}^d} \frac{v_r^0 + v_r^{0,\prime}}{2} \Big\|_1^2 \,\mathrm{d}r\Big) \,|\, \mathcal{F}_t^0\Big], \quad t \in [0, T]
$$

Key functional - I

- Follow Lasry-Lions proof of stability for standard MFGs

$$
\begin{aligned}
&\mathrm{d}_t\big[\varepsilon_1|\delta Y_t|^2 - (\delta u_t, \delta\mu_t)\big] \\
&= \Big[\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + \Big(|\nabla_x \delta u_t|^2, \frac{\mu_t+\mu_t'}{2}\Big) - \Big(\delta\mu_t, \delta Z_t^0 \cdot \frac{v_t^0 + v_t^{0,\prime}}{2}\Big)\Big] \\
&\quad + \mathrm{dmart}_t
\end{aligned}
$$

  - intuitively, needs the $\mathrm{d}t$ term in RHS to be ≥ 0

- Need penalization to kill bad term

$$
e_t = \overline{\mathbb{E}}^0\Big[\exp\Big(-\frac{\sigma_0^2}{A}\int_t^T \Big\| \frac{v_r^0 + v_r^{0,\prime}}{2} - \fint_{\mathbb{T}^d} \frac{v_r^0 + v_r^{0,\prime}}{2} \Big\|_1^2 \mathrm{d}r\Big) \,|\, \mathcal{F}_t^0\Big], \quad t \in [0,T]
$$

  - !!previous BMO estimates!! (still correct after changes of measures)

$$
\frac{1}{A} \ll 1 \Rightarrow e_t^{-1} \leq C(\sigma_0, T)
$$

Key functional - II

- Replace previous functional by

$$\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t, \delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\big|\nabla_x \delta u_r\big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]$$

$$\|\mu - \mu'\|_{-1} = \mathbb{W}_1(\mu, \mu')$$

Key functional - II

- Replace previous functional by

$$\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t,\delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]$$

$$\|\mu-\mu'\|_{-1} = \mathbb{W}_1(\mu,\mu')$$

- Expansion

$$\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t,\delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]$$

$$= \Big[\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x\delta u_t|^2, \frac{\mu_t+\mu_t'}{2}) - \Big(\delta\mu_t, \delta Z_t^0\cdot\frac{v_t^0+v_t^{0,\prime}}{2}\Big)$$

$$+\varepsilon_2\frac{\sigma_0^2}{A}e_t\Big\|\frac{v_t^0+v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x\delta u_r\Big|^2, \frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)$$

$$+\varepsilon_2 e_t\Big(\Big|\nabla_x\delta u_t\Big|^2, \frac{\mu_t+\mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t$$

Key functional - II

- Expansion

$$\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2-(\delta\mu_t,\delta u_t)+\varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2+\int_0^t\Big(\Big|\nabla_x\delta u_r\Big|^2,\frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]$$

$$\geq\Big[\tfrac{1}{2}\varepsilon_1\sigma_0^2|\delta Z_t^0|^2+(|\nabla_x\delta u_t|^2,\frac{\mu_t+\mu_t'}{2})-\frac{1}{\varepsilon_1\sigma_0^2}\|\delta\mu_t\|_{-1}^2\Big\|\frac{v_t^0+v_t^{0,\prime}}{2}\Big\|_1^2$$

$$+\varepsilon_2\frac{\sigma_0^2}{A}e_t\Big\|\frac{v_t^0+v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2+\int_0^t\Big(\Big|\nabla_x\delta u_r\Big|^2,\frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)$$

$$+\varepsilon_2 e_t\Big(\Big|\nabla_x\delta u_t\Big|^2,\frac{\mu_t+\mu_t'}{2}\Big)\Big]\,\mathrm{d}t+\mathrm{dmart}_t$$

Key functional - II

- Expansion

$$
\begin{aligned}
&\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t,\delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]\\
&\geq \Big[\tfrac{1}{2}\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x\delta u_t|^2, \frac{\mu_t+\mu_t'}{2}) - \frac{1}{\varepsilon_1\sigma_0^2}\|\delta\mu_t\|_{-1}^2\Big\|\frac{v_t^0+v_t^{0,\prime}}{2}\Big\|_1^2\\
&\quad + \varepsilon_2\frac{\sigma_0^2}{A}\cancel{e_t}\,c\Big\|\frac{v_t^0+v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t\Big(\Big|\nabla_x\delta u_r\Big|^2, \frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)\\
&\quad + \varepsilon_2 e_t\Big(\Big|\nabla_x\delta u_t\Big|^2, \frac{\mu_t+\mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t
\end{aligned}
$$

Key functional - II

- Expansion

$$\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t, \delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]$$

$$\geq \Big[\frac{1}{2}\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x \delta u_t|^2, \frac{\mu_t + \mu_t'}{2})$$

$$+ \Big(\varepsilon_2 \frac{c\sigma_0^2}{A} - \frac{1}{\varepsilon_1\sigma_0^2}\Big)\Big\|\frac{v_t^0 + v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)$$

$$+ \varepsilon_2 e_t\Big(\Big|\nabla_x \delta u_t\Big|^2, \frac{\mu_t + \mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t$$

Key functional - II

- Expansion

$$
\begin{aligned}
&\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t, \delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big] \\
&\geq \Big[\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x \delta u_t|^2, \frac{\mu_t + \mu_t'}{2}) \\
&\qquad + c\Big\|\frac{v_t^0 + v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big) \\
&\qquad + \varepsilon_2 e_t\Big(\Big|\nabla_x \delta u_t\Big|^2, \frac{\mu_t + \mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t
\end{aligned}
$$

Key functional - II

- Expansion

$$\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t, \delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]$$

$$\geq \Big[\varepsilon_1 \sigma_0^2 |\delta Z_t^0|^2 + (|\nabla_x \delta u_t|^2, \frac{\mu_t + \mu_t'}{2})$$

$$+ c\Big\|\frac{v_t^0 + v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)$$

$$+ \varepsilon_2 e_t\Big(\Big|\nabla_x \delta u_t\Big|^2, \frac{\mu_t + \mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t$$

- Boundary condition
  - monotonicity $\Rightarrow -(\delta u_T, \delta\mu_T) \leq C(|x_0' - x_0|^2 + \|\delta\mu_0\|_{-1}^2) + \frac{1}{4}\mathrm{RHS}$

Key functional - II

- Expansion

$$\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t,\delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t\Big(\Big|\nabla_x\delta u_r\Big|^2,\frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]$$

$$\geq \Big[\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x\delta u_t|^2,\frac{\mu_t+\mu_t'}{2})$$

$$+ c\Big\|\frac{v_t^0+v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t\Big(\Big|\nabla_x\delta u_r\Big|^2,\frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)$$

$$+ \varepsilon_2 e_t\Big(\Big|\nabla_x\delta u_t\Big|^2,\frac{\mu_t+\mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t$$

- Boundary condition
  - monotonicity $\Rightarrow$ $-(\delta u_T,\delta\mu_T) \leq C(|x_0'-x_0|^2 + \|\delta\mu_0\|_{-1}^2) + \frac{1}{4}\mathrm{RHS}$
  - Lipschitz $\Rightarrow$ $\varepsilon_1|\delta Y_T|^2 \leq C\varepsilon_1\|\delta\mu_T\|_{-1}^2 \leq$ " "

Key functional - II

- Expansion

$$\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t,\delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]$$

$$\geq \Big[\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x\delta u_t|^2, \frac{\mu_t+\mu_t'}{2})$$

$$+ c\Big\|\frac{v_t^0+v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t\Big(\Big|\nabla_x\delta u_r\Big|^2, \frac{\mu_r+\mu_r'}{2}\Big)\,\mathrm{d}r\Big)$$

$$+ \varepsilon_2 e_t\Big(\Big|\nabla_x\delta u_t\Big|^2, \frac{\mu_t+\mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t$$

- Boundary condition
  - monotonicity $\Rightarrow$ $-(\delta u_T, \delta\mu_T) \leq C(|x_0'-x_0|^2 + \|\delta\mu_0\|_{-1}^2) + \frac{1}{4}\mathrm{RHS}$
  - Lipschitz $\Rightarrow$ $\varepsilon_1|\delta Y_T|^2 \leq C\varepsilon_1\|\delta\mu_T\|_{-1}^2 \leq$ "   "

Key functional - II

- Expansion

$$
\begin{aligned}
&\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t, \delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big] \\
&\geq \Big[\varepsilon_1\sigma_0^2|\delta Z_t^0|^2 + (|\nabla_x \delta u_t|^2, \frac{\mu_t + \mu_t'}{2}) \\
&\quad + c\Big\|\frac{v_t^0 + v_t^{0,\prime}}{2}\Big\|_1^2\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big) \\
&\quad + \varepsilon_2 e_t\Big(\Big|\nabla_x \delta u_t\Big|^2, \frac{\mu_t + \mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t
\end{aligned}
$$

- Boundary condition
  - ◦ monotonicity $\Rightarrow\ -(\delta u_T, \delta\mu_T) \leq C(|x_0' - x_0|^2 + \|\delta\mu_0\|_{-1}^2) + \frac{1}{4}\mathrm{RHS}$
  - ◦ Lipschitz $\Rightarrow\ \varepsilon_1|\delta Y_T|^2 \leq C\varepsilon_1\|\delta\mu_T\|_{-1}^2 \leq$ " "
- Conclusion: $|\delta Y_0|^2 - (\delta u_0, \delta\mu_0) \leq C(|x_0' - x_0|^2 + \|\delta\mu_0\|_{-1}^2)$

Key functional - II

- Expansion

$$
\begin{aligned}
&\mathrm{d}_t\Big[\varepsilon_1|\delta Y_t^0|^2 - (\delta\mu_t, \delta u_t) + \varepsilon_2 e_t\Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)\Big]\\
&\geq \Big[\varepsilon_1 \sigma_0^2 |\delta Z_t^0|^2 + (|\nabla_x \delta u_t|^2, \frac{\mu_t + \mu_t'}{2})\\
&\quad + c\Big\|\frac{v_t^0 + v_t^{0,\prime}}{2}\Big\|_1^2 \Big(\|\delta\mu_0\|_{-1}^2 + \int_0^t \Big(\Big|\nabla_x \delta u_r\Big|^2, \frac{\mu_r + \mu_r'}{2}\Big)\,\mathrm{d}r\Big)\\
&\quad + \varepsilon_2 e_t \Big(\Big|\nabla_x \delta u_t\Big|^2, \frac{\mu_t + \mu_t'}{2}\Big)\Big]\,\mathrm{d}t + \mathrm{dmart}_t
\end{aligned}
$$

- Boundary condition
  - monotonicity $\Rightarrow\ -(\delta u_T, \delta\mu_T) \leq C(|x_0' - x_0|^2 + \|\delta\mu_0\|_{-1}^2) + \frac{1}{4}\mathrm{RHS}$
  - Lipschitz $\Rightarrow\ \varepsilon_1|\delta Y_T|^2 \leq C\varepsilon_1\|\delta\mu_T\|_{-1}^2 \leq$ " "
- Conclusion: $|\delta Y_0|^2 + \|\delta u_0\|_s^2 \leq C(|x_0' - x_0|^2 + \|\delta\mu_0\|_{-1}^2)$

Thank you!