Functional Convergence of Berry's Nodal Lengths

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Berry's Random Waves (1977)

- **Berry's planar random wave**, written

$$B_E = \left\{ B_E(x) : x \in \mathbb{R}^2 \right\}, \qquad E > 0$$

is the unique planar centred, isotropic Gaussian field such that

$$\Delta B_E + 4\pi^2 E \cdot B_E = 0 \quad \text{a.s.} \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

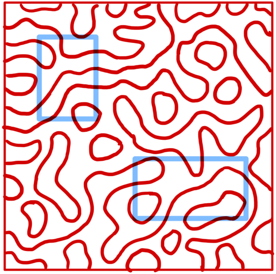
- Equivalently,

$$\mathbb{E}\left[B_E(x) B_E(y)\right] := J_0(2\pi\sqrt{E}\|x-y\|)$$

- Write $b = \left\{ b(x) : x \in \mathbb{R}^2 \right\}$ for $B_{(4\pi^2)^{-1}}$

- Think of b as a "canonical" Gaussian Laplace eigenfunction on $\mathbb{R}^2$, emerging e.g. as a universal local scaling limit for arithmetic and monochromatic RWs, random spherical harmonics ...
  See: *Marinucci & Rossi (2016), Canzani and Hanin (2021), Dierickx, Nourdin, Peccati & Rossi (2023)*.

- $B_E^{-1}(0) := \{x \in \mathbb{R}^2 : B_E(x) = 0\}$ smooth curves
- $\mathscr{L}_E(Q) :=$ length $\left(B_E^{-1}(0) \cap Q\right)$ rectangle $Q \subset \mathbb{R}^2$



GOAL

To fix ideas, let $\mathcal{Q}$ be the collection of all rectangles $Q \subset [0,1]^2$.

- Forevery $E \geq 1$ and $Q \in \mathcal{Q}$, consider $\mathscr{L}_E(Q)$.
- **Task 1**: describe the joint fluctuations, as $E \to \infty$, of the random variables $\mathscr{L}_E(Q)$, $Q \in \mathcal{Q}$.
- **Task 2**: describe the functional fluctuations, as $E \to \infty$, of the random function

$$(s_1, s_2) \mapsto \mathscr{L}_E(s_1, s_2) := \mathscr{L}_E([0, s_1] \times [0, s_2]) \qquad s_1, s_2 \in [0,1]$$

- *Berry (J. Phys. A, 2002)* – as $E \to \infty$:

$$\mathbb{E}\left[\mathscr{L}_E(Q)\right] = \frac{\pi \ \text{area}\, Q}{\sqrt{2}} \sqrt{E} \qquad \text{Var}\left(\mathscr{L}_E(Q)\right) \sim \frac{\text{area}\, Q}{512\pi} \log E$$

- Such an estimate follows from an analytical cancellation in Kac-Rice formulae: the natural guess for the order of the variance is $\sqrt{E}$.

- *Nourdin, Peccati & Rossi (CMP, 2019)*:

$$\sqrt{\frac{512\pi}{\log E}} \big(\mathscr{L}_E(Q) - \mathbb{E}\mathscr{L}_E(Q)\big) \xrightarrow{d} \mathcal{N}(0, \text{area}(Q)),$$

Define:

$$\widetilde{\mathscr{L}}_E(Q) := \sqrt{\frac{512\pi}{\log E}}\,\{\mathscr{L}_E(Q) - \mathbb{E}\,[\mathscr{L}_E(Q)]\}\,, \quad t \geq 1\,,$$

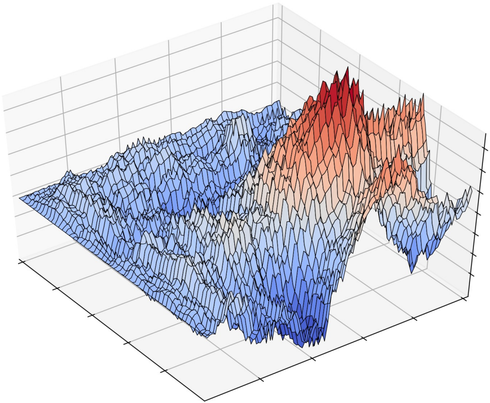
and similarly $\left\{\widetilde{\mathscr{L}}_E(s_1, s_2) : (s_1, s_2) \in [0,1]^2\right\}$.

1. For all $Q_1, \ldots, Q_d \in \mathcal{Q}$, as $E \to \infty$, $\left(\widetilde{\mathscr{L}}_E(Q_1), \ldots, \widetilde{\mathscr{L}}_E(Q_d)\right)$ converges to a centered Gaussian vector with covariance function $\Sigma(i,j) = \text{area}(Q_i \cap Q_j)$.

2. As $E \to \infty$, the random field $\left\{\widetilde{\mathscr{L}}_E(s_1, s_2) : (s_1, s_2) \in [0,1]^2\right\}$ converges in the f.d.d.-sense to a standard Wiener sheet.

A standard Wiener sheet $\left\{\mathbf{W}(\mathbf{s}), \mathbf{s} \in [0,1]^2\right\}$ is a centred Gaussian process with covariance

$$\mathbb{E}\left[\mathbf{W}(s_1, s_2)\mathbf{W}(t_1, t_2)\right] = (s_1 \wedge t_1)(s_2 \wedge t_2)$$



A realization of a Wiener sheet, pic by George Lowther

Question: Does $\left\{\widetilde{\mathscr{L}}_E(s_1, s_2)\right\}$ converge to a Wiener sheet *as a random function* (i.e. in $\mathcal{D}_2$, the Skorohod space of cadlag mappings on $[0,1]^2$)?

Lemma. $\{X, X_n : n \geq 1\} \subset \mathcal{D}_2, \quad X_n = U_n + V_n + W_n$

(a) as $n \to \infty$, U_n converges weakly to X in $\mathcal{D}_2$,

(b) as $n \to \infty$, V_n converges weakly to zero in $\mathcal{D}_2$,

(c) for every $\varepsilon > 0$, $\lim_{n \longrightarrow \infty} \mathbb{P}\left\{\sup_{\mathbf{t} \in [0,1]^2} |W_n(\mathbf{t})| > \varepsilon\right\} = 0$,

$\Longrightarrow$ X_n converges weakly to X in $\mathcal{D}_2$.

- $\widetilde{\mathscr{L}}_E[q] := \text{proj}(\widetilde{\mathscr{L}}_E | C_q)$, $\quad C_q$:=the qth Wiener chaos associated with b.
- Then,

$$\widetilde{\mathscr{L}}_E = \widetilde{\mathscr{L}}_E[2] + \widetilde{\mathscr{L}}_E[4] + R_E, \quad \text{where} \quad R_E := \sum_{q\geq 3} \widetilde{\mathscr{L}}_E[2q]\,,$$

Strategy: applying the previous lemma to

$$(X_n, U_n, V_n, W_n) = (\widetilde{\mathscr{L}}_E, \widetilde{\mathscr{L}}_E[4], \widetilde{\mathscr{L}}_E[2], R_E)$$

(I) $\widetilde{\mathscr{L}}_E[4]$ converges weakly to a standard Wiener sheet (OK – easy);

(II) $\widetilde{\mathscr{L}}_E[2]$ converges weakly to zero;

(III) the residual term R_E converges uniformly to zero in probability.

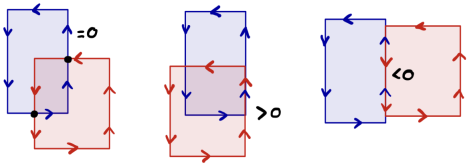
As $E \to \infty$,

$$\mathbf{Cov}\left(\mathscr{L}_E[2](Q_1), \mathscr{L}_E[2](Q_2)\right) = \frac{\lambda(\partial Q_1, \partial Q_2)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right),$$

where

$$\lambda(\partial Q_1, \partial Q_2) = \int_{\partial Q_1 \cap \partial Q_2} \langle \mathrm{n}_1(x), \mathrm{n}_2(x)\rangle \, d\mathcal{H}^1(x).$$

indicates the **signed length** of $\partial Q_1 \cap \partial Q_2$.



- As $E \to \infty$,

$$4\pi\, E^{1/4}\Big(\mathscr{L}_E[2](Q_1), ..., \mathscr{L}_E[2](Q_d)\Big) \xrightarrow{d} \mathcal{N}_d(0, \Sigma)$$

$$\Sigma(i,j) = \lambda(\partial Q_i, \partial Q_j)$$

- $\mathbf{Var}\big(\mathscr{L}(b; R\cdot Q)[2]\big) = \dfrac{R \cdot \text{length}(\partial Q)}{8\pi} + o(R) \qquad R = 2\pi\sqrt{E}$

Step (II) – A connection with Buckley & Sodin (2017)

- **Gaussian entire function**: $\{\zeta_n\}$ i.i.d. complex std Gaussian

$$z \mapsto f(z) = \sum_{n=0}^{\infty} \zeta_n \frac{z^n}{n!} \qquad z \in \mathbb{C}$$

- $f_R(z) := f(Rz) \qquad n_R(Q) := \#\left\{f_R^{-1}(0) \cap Q\right\}$

- $\mathbf{Var}\,(n_R(Q)) = c_0\, R \cdot \text{length}\,(\partial Q) + o(R) \qquad R \to \infty$

- $\frac{1}{\sqrt{c_0\, R}}\left((n_R(Q_1) - \mathbb{E}\left[n_R(Q_1)\right]), ..., (n_R(Q_d) - \mathbb{E}\left[n_R(Q_d)\right])\right) \xrightarrow{d} \mathcal{N}_d(0, \Sigma)$

- *hyperuniformity*: a variance that scales as the length of the boundary of $R \cdot Q$, rather than as $\mathrm{area}(R \cdot Q) \asymp R^2$.

- *total disorder process*:
  - the linear span of the processes $\mathscr{L}_E(Q)[2]$ and $n_E(Q)$ contains an uncountable collection of i.i.d. centered Gaussian random variables with unit variance (e.g. $\mathscr{L}_E(\mathbf{s})[2]$).
  - physics, random matrix theory.

- As $E \to \infty$, the field $\left\{ \widetilde{\mathscr{L}_E}(\mathbf{s})[2] : (\mathbf{s}) \in [0,1]^2 \right\}$ weakly converges to zero in $\mathcal{D}_2$ (tightness + estimates for sup of stationary Gaussian fields).

Fix $K \geq 1$, we define the partition Π_K of $[0,1]^2$ formed by the collection of squares of side length 2^{-K}:

- For every $i = (i_1, i_2) \in \left\{0, \ldots, 2^K\right\}^2$, we define the *partition points* $\mathbf{p}_i(K,K) := (p_{i_1}(K), p_{i_2}(K)) \in [0,1]^2$ by

$$p_{i_1}(K) := \frac{i_1}{2^K}, \quad p_{i_2}(K) := \frac{i_2}{2^K}, \quad i_1, i_2 = 0, 1, \ldots, 2^K.$$

- For $\mathbf{s} = (s_1, s_2) \in [0,1]^2$, we write $i_{K,K}(\mathbf{s}) = (i_{1,K}(s_1), i_{2,K}(s_2))$ for the vector verifying

$$p_{i_{1,K}(s_1)} \leq s_1 < p_{i_{1,K}(s_1)+1} \qquad p_{i_{2,K}(s_2)} \leq s_2 < p_{i_{2,K}(s_2)+1}$$

  that is, the vector $i_{K,K}(\mathbf{s})$ is such that $\mathbf{p}_{i_{K,K}(\mathbf{s})}(K,K)$ is the closest partition point to s on the left.

Discretized nodal length:

$$\mathscr{L}_E^K(s_1, s_2) := \mathscr{L}_E\left([0, p_{i_{1,K}(s_1)}(K)] \times [0, p_{i_{2,K}(s_2)}(K)]\right)$$

Take $\{K(E) : E > 0\}$ numerical sequence s.t. $K(E) \to \infty$ and $K(E) = o((\log E)^{1/10})$ as $E \to \infty$. Then,

1. for every $\varepsilon > 0$, $\quad \mathbb{P}\left\{\sup_{\mathbf{s}\in[0,1]^2} \left|R_E^{K(E)}(\mathbf{s})\right| > \varepsilon\right\} \longrightarrow 0$
2. the normalized process $\left\{\widetilde{\mathscr{L}}_E^{K(E)}(\mathbf{s})\right\}$ converges weakly to a standard Wiener sheet $\mathbf{W}$ on $[0,1]^2$ in the Skorohod space $\mathcal{D}_2$

Main difficulty for directly dealing with the residual term R_E:

- The expectation of $\mathscr{L}_E(\mathbf{s})$ (order $\sqrt{E/\log E}$) grows much faster than the normalizing factor $\log E$.

- Planar chaining argument with R_E requires

$$\left|\mathbb{E}\left[\mathscr{L}_E(\mathbf{t})\right] - \mathbb{E}\left[\mathscr{L}_E(\mathbf{p}_{i_{K,K}(\mathbf{t})}(K,K))\right]\right| \approx \frac{\sqrt{E}}{\sqrt{\log E}}\frac{1}{2^K}$$

  to be bounded.

- This requirement is incompatible with the constrained choice $K(E) = o((\log E)^{1/10})$, as is needed in the above statements.

Thank you!