Reflected BSDE driven by a marked point process with a convex/concave generator

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Reflected BSDEs (RBSDEs)

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s dB_s + K_T - K_t; \\ Y_t \geq L_t, \quad \forall t \in [0, T]. \end{cases} \tag{1}$$

- K is an increasing process, satisfying Skorokhod condition:

$$\int_0^T (Y_t - L_t)\,dK_t = 0.$$

- El Karoui et al., 1997: arised from pricing of American contingent claims, Lipschitz generator+square integarble terminal;
- Matoussi, 1997: linear growth in (y, z)+square integarble terminal;
- Kobylanski et al., 2002: superlinear in y and quadratic in z +bounded terminal and obstacle;
- Lepeltier and Xu, 2007, Bayraktar and Yao, 2012: unbounded terminal.

Reflected BSDEs with jump (RBSDEJs)

$$
\begin{cases}
Y_t = \xi + \int_t^T f(s, Y_s, U_s(\cdot))\, ds + \int_t^T dK_s - \int_t^T \int_E U_s(e) q(dsde), \quad 0 \le t \le T; \\
Y_t \ge L_t, \quad 0 \le t \le T; \\
\int_0^T (Y_{s^-} - L_{s^-})\, dK_s = 0.
\end{cases}
\tag{2}
$$

- $\tilde{q} = p - \nu$: martingale, (e.g. compensated Poisson process); L: barrier: càdlàg. .
- f: Lipschitz in (y, z, u).
- Generlization in jump process: Hamadène and Ouknine, 2016; Hamadène and Ouknine, 2003: Poisson process; Ren and Otmani, 2010: Lévy process; Crépey and Matoussi, 2008: MPP with bounded density; Foresta, 2021: general MPP.
- Generlization in barrier: Grigorova et al., 2017, 2020: optional process.

Quadratic-Exponential BSDEJs

Generalization in growth condition: Quadratic-Exponential growth:

$$-\frac{\lambda}{2}|z|^2 - \alpha_t - \beta|y| - \frac{1}{\lambda}j_\lambda(t,-u) \le f(t,y,z,u) \le \frac{1}{\lambda}j_\lambda(t,u) + \alpha_t + \beta|y| + \frac{\lambda}{2}|z|^2, \quad (3)$$

- $j_\lambda(t,u) = \int_E (e^{\lambda u(e)} - \lambda u(e) - 1)\phi_t(de)$, $\nu(\omega,dt,dx) = \phi_t(\omega,de)dt$.
- Related to the quadratic variation of the Doléans-Dade exponential of the solution.
- BSDEs with bounded terminal: monotone: Becherer, 2006, Morlais, 2010; fixed point: Kazi-Tani et al., 2015.
- Semimartingale viewpoint: Ngoupeyou, 2010, Jeanblanc et al., 2016, El Karoui et al., 2016, existence for unbounded terminal.
- Application in exponential uitility maximization in jump market
- Particular form of generator related to utility maximization problem: Kaakai et al., 2022.
- Generalization of classical quadratic BSDEs: Kobylanski, 2000

Our results: wellposedness of RBSDEs driven by MPP

For simplicity, not involve z, with the help of Briand and Hu, 2006, 2008; Delbaen et al., 2011, 2015 to generalize.

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, U_s)\, dA_s + \int_t^T dK_s - \int_t^T \int_E U_s(e) q(dsde), \quad 0 \le t \le T; \\ Y_t \ge L_t, \quad 0 \le t \le T; \\ \int_0^T (Y_{s-} - L_{s-})\, dK_s = 0. \end{cases} \tag{4}$$

- MPP associated random discrete measure p on $((0, +\infty) \times E, \mathscr{B}((0, +\infty) \times E))$:

$$p(\omega, D) = \sum_{n \ge 1} \mathbf{1}_{(T_n(\omega), \zeta_n(\omega)) \in D}, \quad \forall \omega \in \Omega. \tag{5}$$

- $q(dtde) := p(dtde) - \phi_t(de) dA_t$: martingale measure; process A may not absolutely continuous (See e.g. Janson et al., 2011).
- Wellposedness for unbounded terminal with convex / concave generators.

Marked point process

- $(\Omega, \mathscr{F}, \mathbb{P})$: a complete probability space
- E: mark space equipped with the Borel σ-algebra $\mathscr{B}(E)$.
- Given a sequence of random variables (T_n, ζ_n) taking values in $[0, \infty] \times E$, set $T_0 = 0$ and $\mathbb{P} - a.s.$
  1. $T_n \leq T_{n+1}, \ \forall n \geq 0$;
  2. $T_n < \infty$ implies $T_n < T_{n+1}$ $\forall n \geq 0$.
- $(T_n, \zeta_n)_{n \geq 0}$: marked point process (MPP).
- We assume the marked point process is non-explosive, i.e., $T_n \to \infty$, $\mathbb{P} - a.s.$.
- MPP associated random discrete measure p on $((0, +\infty) \times E, \mathscr{B}((0, +\infty) \times E))$:

$$p(\omega, D) = \sum_{n \geq 1} \mathbf{1}_{(T_n(\omega), \zeta_n(\omega)) \in D}, \ \forall \omega \in \Omega. \tag{6}$$

Assumptions

(H1) The process A is continuous, with $\|A_T\|_\infty < \infty$. $\Rightarrow$ The MPP is totally inaccessible.

(H2) The obstacle process L is continuous with $L_T \leq \xi$. $\Rightarrow$ The process K is continuous.

(H3) For every $\omega \in \Omega$, $t \in [0, T]$, $r \in \mathbb{R}$, the mapping $f(\omega, t, r, \cdot) : L^0(\mathcal{B}(E)) \to \mathbb{R}$ satisfies: for every $U \in H^{2,2}_\nu$,

$$(\omega, t, r) \mapsto f\left(\omega, t, r, U_t(\omega, \cdot)\right)$$

is Prog $\otimes \mathscr{B}(\mathbb{R})$-measurable.

Assumption cont.

(H4) (a) (Continuity condition) For every $\omega \in \Omega,\ t \in [0,T],\ y \in \mathbb{R},\ u \in L^2(E, \mathcal{B}(E), \phi_t(\omega, dy))$, $(y,u) \longrightarrow f(t,y,u)$ is continuous.

(b) (Lipschitz condition in y) There exist $\tilde{\beta} \geq 0$, such that for every $\omega \in \Omega,\ t \in [0,T],\ y, y' \in \mathbb{R},\ u \in L^2(E, \mathcal{B}(E), \phi_t(\omega, dy))$, we have

$$|f(\omega,t,y,u(\cdot)) - f(\omega,t,y',u(\cdot))| \leq \tilde{\beta}\,|y - y'|\,.$$

(c) (Quadratic-exponential growth condition) For all $t \in [0,T]$, $(y,u) \in \mathbb{R} \times L^2(E, \mathcal{B}(E), \phi_t(\omega, dy))$: $\mathbb{P}$-a.s,

$$\underline{q}(t,y,u) = -\frac{1}{\lambda} j_\lambda(t,-u) - \alpha_t - \beta|y| \leq f(t,y,u) \leq \frac{1}{\lambda} j_\lambda(t,u) + \alpha_t + \beta|y| = \bar{q}(t,y,u).$$

where $\{\alpha_t\}_{0 \leq t \leq T}$ is a progressively measurable nonnegative stochastic process. $j_\lambda(t,u) = \int_E (e^{\lambda u(e)} - \lambda u(e) - 1)\phi_t(de)$.

Assumption cont.

(d) (Integrability condition) We assume necessarily,

$$\forall p > 0, \quad \mathbb{E}\left[\exp\left\{p\lambda e^{\beta A_T}(|\xi| \vee L_*^+) + p\lambda \int_0^T e^{\beta A_s}\alpha_s dA_s\right\}\right] < \infty.$$

(e) (Convexity/Concavity condition) For each $(t, y) \in [0, T] \times \mathbb{R}$, $u \in L^2(E, \mathcal{B}(E), \phi_t(\omega, dy))$, $u \to f(t, y, u)$ is convex or concave.

Bounded Lipschitz case

Assume additionally:

(H4') (a) There exists a constant $M > 0$ such that $L_* + |\xi| \leq M$, $\mathbb{P} - a.s.$

(b) There exist $L_f \geq 0$, $L_U \geq 0$ such that for every $\omega \in \Omega$, $t \in [0,T]$, $y, y' \in \mathbb{R}$, $u, u' \in L^2(E, \mathcal{B}(E), \phi_t(\omega, dy))$ we have

$$|f(\omega,t,y,u(\cdot)) - f(\omega,t,y',u'(\cdot))| \leq L_f |y-y'| + L_U \left(\int_E |u(e)-u'(e)|^2 \phi_t(\omega,de)\right)^{1/2}.$$

(c) We have

$$\mathbb{E}\left[\int_0^T |f(s,0,0)|^2 dA_s\right] < \infty.$$

Bounded Lipchitz case

Theorem 1

Let assumptions (H1), (H2), (H3) and (H4') hold, then,
(i) there exists a solution $(Y, U, K) \in L^2(A) \times H_\nu^{2,2} \times \mathbb{K}^2$ *to (4). Moreover, with the help of Foresta, 2021, Lemma 3.2,* $Y \in S^2$.
(ii) If in addition, there exists a positive constant M_0 *such that , for each* $t \in [0, T]$ *and* $(y, u) \in \mathbb{R} \times L^2(E, \mathcal{B}(E), \phi_t(\omega, dy))$

$$|f(t, y, u)| \leq M_0.$$

Then, there exists a unique solution $(Y, U, K) \in S^\infty \times J^\infty \times \mathbb{K}^2$.

Need better uniform estimations in the sequel. $H_\nu^{2,p}$ is the space of predictable processes U such that $\|U\|_{H_\nu^{2,p}} := \left(\mathbb{E}\left[\int_{[0,T]}\int_E |U_s(e)|^2 \phi_s(de) dA_s\right]^{\frac{p}{2}}\right)^{\frac{1}{p}} < \infty$.

Comparison theorem

Theorem 2

Let $(\xi, f, L), (\hat{\xi}, \hat{f}, \widehat{L})$ *be two parameter sets and let* $(Y, U, K)($ *resp.* $(\widehat{Y}, \widehat{U}, \widehat{K}))$ *be a solution of* $RBSDE(\xi, f, L)$ *(resp.* $RBSDE(\hat{\xi}, \hat{f}, \widehat{L})$*) such that* $\mathbb{P}$*-a.s.,* $\xi \leq \hat{\xi}$ *and that* $L_t \leq \widehat{L}_t$ *for any* $t \in [0, T]$*. For process* α *and constants* $\beta, \tilde{\beta} \geq 0, \lambda > 0$*, suppose (H1)-(H2) and (H4)(d) hold,* $Y,\ \hat{Y} \in \mathcal{E}$*,* $U,\ \hat{U} \in H_\nu^{2,p}$ *and* $K,\ \hat{K} \in \mathbb{K}^p$*, for each* $p \geq 1$*. If in addition either of the following two holds:*

(i) f *satisfies (H3), (H4)(a-b),* f *is convex in* u*,* $\Delta f(t) := f\left(t, \widehat{Y}_t, \widehat{U}_t\right) - \hat{f}\left(t, \widehat{Y}_t, \widehat{U}_t\right) \leq 0$*,* $dt \otimes d\mathbb{P}$*-a.e., and* $f(t, y, u) \leq \alpha_t + \beta|y| + \frac{1}{\lambda} j_\lambda(t, u)$*;*

(ii) $\hat{f}$ *satisfies (H3), (H4)(a-b),* $\hat{f}$ *is convex in* u*,*
$\Delta f(t) := f(t, Y_t, U_t) - \hat{f}(t, Y_t, U_t) \leq 0$*,* $dt \otimes d\mathbb{P}$*-a.e., and*
$\hat{f}(t, y, u) \leq \alpha_t + \beta|y| + \frac{1}{\lambda} j_\lambda(t, u)$*;*

then it holds $\mathbb{P}$*-a.s. that* $Y_t \leq \widehat{Y}_t$ *for any* $t \in [0, T]$*.*

Comparison theorem

- The θ-method; Fix $\theta \in (0,1).$, set $\tilde{Y} := Y - \theta\widehat{Y}, \tilde{U} := U - \theta\widehat{U}$;
- To deal with j term: define

$$a_t := \mathbf{1}_{\{Y_t \geq 0\}} \left(\mathbf{1}_{\{Y_t \neq \widehat{Y}_t\}} \frac{\mathfrak{F}\left(t, Y_t, \widehat{U}_t\right) - \mathfrak{F}\left(t, \widehat{Y}_t, \widehat{U}_t\right)}{Y_t - \widehat{Y}_t} - \tilde{\beta}\mathbf{1}_{\{Y_t = \widehat{Y}_t\}} \right) - \tilde{\beta}\mathbf{1}_{\{Y_t < 0 \leq \widehat{Y}_t\}}$$

$$+ \mathbf{1}_{\{Y_t \vee \widehat{Y}_t < 0\}} \left(\mathbf{1}_{\{\tilde{Y}_t \neq 0\}} \frac{\mathfrak{F}(t, Y_t, U_t) - \mathfrak{F}\left(t, \theta\widehat{Y}_t, U_t\right)}{\tilde{Y}_t} - \tilde{\beta}\mathbf{1}_{\{\tilde{Y}_t = 0\}} \right), \quad t \in [0,T],$$

and $\tilde{A}_t := \int_0^t a_s dA_s, t \in [0,T]$, estimate the exponential transform $\Gamma_t := \exp\left\{\zeta_\theta e^{\tilde{A}_t}\tilde{Y}_t\right\}$, where $\zeta_\theta := \frac{\lambda e^{\tilde{\beta}\|A_T\|_\infty}}{1-\theta}$.

Comparison theorem

- Plug in suitable estimations for K.
- $$\begin{aligned} Y_t - \theta \widehat{Y}_t \leq & \frac{1-\theta}{\lambda} \ln\left(1 \vee \frac{\lambda e^{\tilde{\beta}\|A_T\|_\infty}}{1-\theta}\right) e^{-\tilde{\beta}\|A_T\|_\infty - \tilde{A}_t} \\ & + \frac{1-\theta}{\lambda}\left(e^{\tilde{\beta}\|A_T\|_\infty} + \ln\left(\mathbb{E}\left[\eta\left(1+K_T\right) \mid \mathcal{G}_t\right]\right)\right) e^{-\tilde{\beta}\|A_T\|_\infty - \tilde{A}_t}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$
- We get rid of A_γ condition: $\gamma > -1$,

$$f(t,y,z,u) - f\left(t,y,z,u'\right) \leq \int_E \gamma_t(x)\left(u-u'\right)(x)\psi_t(dx).$$

A priori estimates

Definition 3 (Solution to the RBSDE)

Under assumptions (H1)-(H4), a solution to RBSDE (4) is a triple process (Y, U, K) on $[0, T]$, in which Y is a càdlàg process, and U is an $\mathbb{G}$-predictable random field. Moreover, for each $p \geq 1$, processes $\int_0^\cdot \int_E (e^{p\lambda U_t(e)} - 1) q(dsde)$ and $\int_0^\cdot \int_E (e^{p\lambda U_t(e)} - 1) q(dsde)$ are local martingales on $[0, T]$. K is a continuous incresing process.

The following additional assumption helps provide uniform estimates for the solutions.

(H5) (Uniform linear bound condition) There exists a positive constant C_0 such that for each $t \in [0, T]$, $u \in L^2(E, \mathcal{B}(E), \phi_t(\omega, dy))$, if f is convex (resp. concave) in u, then $f(t, 0, u) - f(t, 0, 0) \geq -C_0 \|u\|_t$ (resp. $f(t, 0, u) - f(t, 0, 0) \leq C_0 \|u\|_t$).

Weaker than the A_γ condition.

A priori estimates for Y

Bounded obstacle and terminal first (+ monotone convergence⇒ unbounded).

Proposition 4

Let (ξ, f, L) be a parameter set such that (H1'), (H2)-(H3), (H4)(b-e), (H4')(a) and (H5) hold. If $(Y, U, K) \in \mathcal{E} \times H^{2,p} \times \mathbb{K}^p$, for each $p \geq 1$, is a solution of the quadratic exponential $RBSDE(\xi, f, L)$, then it holds $\mathbb{P}$-a.s. that for each $t \in [0, T]$,

$$\exp\{p\lambda |Y_t|\} \leq \mathbb{E}_t \left[\exp\left\{ p\lambda e^{\beta A_T} (|\xi| \vee L_*^+) + p\lambda \int_t^T e^{\beta A_s} \alpha_s dA_s \right\} \right]. \tag{7}$$

By estimations for BSDEs (see our work Gu et al., 2024) and Snell envolope.

A priori estimates for U and K

Proposition 5

Let (ξ, f, L) be a parameter such that (H1)-(H4) hold. If (Y, U, K) is a solution of the quadratic RBSDE(ξ, F, L) such that $Y \in \mathcal{E}$, then for each $p \geq 1$,

$$\mathbb{E}\left[\left(\int_0^T \int_E |U_t(e)|^2 \phi_t(de)dA_t\right)^{\frac{p}{2}} + K_T^p\right] \leq C_p \mathbb{E}\left[e^{36p\lambda(1+\beta\|A_T\|_\infty)Y_*}\right] < \infty, \qquad (8)$$

and also

$$\mathbb{E}\left[\int_0^T \int_E \left(e^{p\lambda|U_t(e)|} - 1\right)^2 \phi_t(de)dA_t\right] \leq C_p \mathbb{E}\left[e^{36p\lambda(1+\beta\|A_T\|_\infty)Y_*}\right] < \infty, \qquad (9)$$

where C_p is a constant depending on p and the constants in (H1)-(H4).

A priori estimates for U and K

1. Estimate the quadratic variation of $\underline{G}_t = -Y_t + \int_0^t \alpha_s dA_s + \int_0^t \beta |Y_s| dA_s$ via Garcia-Neveu Lemma.
2. Estimate $\mathbb{E}[K_T^2]$ via Burkholder-Davis-Gundy inequality. Not easy to obtain $\mathbb{E}[K_T^p]$ directly without adequate integarbility on U at this stage.
3. Define similarly $\bar{G}_t = Y_t + \int_0^t \alpha_s dA_s + \int_0^t \beta |Y_s| dA_s$, estimate the quadratic variation of $e^{p\lambda \bar{G}_t}$ via Garcia-Neveu Lemma. Need the previous esimation on K.
4. Obtain adequate integrability for U from 1 and 3.
5. Estimate $\mathbb{E}[K_T^p]$ via a generalized Burkholder-Davis-Gundy inequality (see Hernández-Hernández and Jacka, 2022, Theorem 2.1).

Existence

Theorem 6 (Existence)

Assume that assumptions (H1)-(H5) are fulfilled. Then the RBSDE (4) admits a unique solution $(Y, U, K) \in \mathcal{E} \times H^{2,p} \times \mathbb{K}^p$*, for all* $p \geq 1$.

- $\mathcal{E}^p$: $e^{|Y|} \in S^p$. Denote $Y \in \mathcal{E}$ if $Y \in \mathcal{E}^p$ for any $p \geq 1$.
- $H_\nu^{2,p}$: predictable processes U such that

$$\|U\|_{H_\nu^{2,p}} := \left(\mathbb{E} \left[\int_{[0,T]} \int_E |U_s(e)|^2 \, \phi_s(de) dA_s \right]^{\frac{p}{2}} \right)^{\frac{1}{p}} < \infty.$$

- $\mathbb{K} \subset \mathbb{C}^0$: increasing and continuous adapted process starting from 0, $\mathbb{K}^p \subset \mathbb{K}$: $X \in \mathbb{K}^p \Leftrightarrow X_T \in \mathbb{L}^p$.
- Begin with bounded case and then generalize to unbounded case.

Existence–bounded terminal and obstacle

We use the following auxiliary generators to approximate convex f.

$$f^n(t,y,u) = \inf_{r \in L^2(E,\mathcal{B}(E),\phi_t(\omega,dy))} \left\{ f(t,y,r) + n\|u-r\|_t \right\}.$$

The properties of the auxiliary drivers are outlined below. See also Lepeltier and Martin, 1997.

Lemma 7

Under the assumptions (H1)–(H4),
(i) The sequence $\{f^n\}_n$ is globally Lipschitz with respect to (y, u) in $\mathbb{R} \times L^2(E,\mathcal{B}(E),\phi_t(\omega,dy))$.
(ii) The sequence $\{f^n\}_n$ is convex with respect to u if f is convex with respect to u for $u \in L^2(E,\mathcal{B}(E),\phi_t(\omega,dy))$.
(iii) For $t \in [0,T]$, the sequence $\{f^n\}_n$ converges to f on $(y,u) \in \mathbb{R} \times L^2(E,\mathcal{B}(E),\phi_t(\omega,dy))$.
Also strong convergence on bounded subsets.

Existence–bounded terminal and obstacle

Lemma 7 cont.

(iv) For $n > C_0$, (Here we need the linear lower bound assumption for the lower bound.)

$$
\begin{aligned}
&-3\alpha_t - 3\beta|y| - \frac{1}{\lambda}j_\lambda(t,-u) \le f^n(t,y,u) \\
&\le f(t,y,u) \le \alpha_t + \beta|y| + \frac{1}{\lambda}j_\lambda(t,u) \le 3\alpha_t + 3\beta|y| + \frac{1}{\lambda}j_\lambda(t,u).
\end{aligned} \tag{10}
$$

(v) For each $n > C_0$ and $t \in [0,T]$, $f^n(t,0,0) = f(t,0,0)$.

Existence–bounded terminal and obstacle

The proof consists of 5 steps. We start from RBSDE $(\xi,\ f^{n,k},\ L)$, where $n > C_0$, $f^{n,k} = (f^n \wedge -k) \vee k$, $k \in \mathbb{N}$. $(Y^{n,k}, U^{n,k}, K^{n,k}) \in S^\infty \times \mathbb{J}^\infty \times \mathbb{K}^2$.

Step 1 The convergence of the sequence $\{(Y^{n,k}, U^{n,k})\}_k$ to (Y^n, U^n).

- For fixed $n > C_0$, $\lim_{k\to\infty} \mathbb{E}\left[|Y_t^n - Y_t^{n,k}|^2\right] = 0$, $\lim_{k\to\infty} \mathbb{E}\left[\int_0^T \int_E |U^{n,k} - U^n|^2 \phi_t(de) dA_t\right] = 0$.
- Stochastic (pointwise) Gronwall inequality involved a martingale term (inspired by Scheutzow, 2013, Theorem 4.)
- To obtain uniform a priori estimates for RBSDE$(\xi,\ f^n,\ L)$.

Step 2 Construction of candidate solution (Y^0, U^0).

- For Y^0, comparison theorem.
- For U^0, stability of Cauchy sequence.
- Based on uniform a priori estimates.
- Y^0 not for sure càdlàg, only a limit for strong convergence in constructing U^0.

Existence–bounded terminal and obstacle

Step 3 A priori estimate of $|Y^n - Y^m|$.

- The θ-method.
- $$|Y_t^n - Y_t^m| \leq (1-\theta)\left(|Y_t^m| + |Y_t^n|\right) + \frac{1-\theta}{\lambda} \ln\left(\sum_{i=1}^{3} J_t^{m,n,i}\right), \quad t \in [0,T],$$

  $J^{m,n,i}$ uniformly bounded in appropriate spaces.

Step 4 Convergence of the sequence $\{Y^n\}$ in S^1.

- Find a càdlàg candidate $\tilde{Y}^0$.
- Need the estimate in step 3.

Step 5 Find candidate solution K^0 and verify the solution $(\tilde{Y}^0, U^0, K^0)$.

- For K^0, Cauchy sequence in S^2.

Existence–unbounded terminal and obstacle

Approximate by solutions in bounded case, i.e., RBSDE(ξ^n, $\bar{f}^n$, L^n) with solution $(Y^n, U^n, K^n) \in \mathcal{E} \times \cap_{p\geq 1} H_\nu^{2,p} \times \cap_{p\geq 1} \mathbb{K}^p$. Here, $\xi^n = (\xi \wedge n) \vee -n$, $L^n = (L \wedge n) \vee -n$, $\bar{f}^n(t,\cdot,\cdot) = f(t,\cdot,\cdot) - f(t,0,0) + f^n(t,0,0)$, $f^n(t,0,0) = (f(t,0,0) \wedge n) \vee -n$.

Step 1 Construction of candidate solution Y^0.

- Find an a priori estimate of $|Y^n - Y^m|$ (solutions for truncated RBSDEs) via the θ-method.
- Some tricks when seperating different obstacles.

Step 2 Construction of candidate solution U^0.

- Cauchy sequence in $H_\nu^{2,2}$.
- A prirori estimates in $H_\nu^{2,p}$.

Step 3 Construction of candidate solution K^0 and verification of the solution (Y^0, U^0, K^0).

- Cauchy sequence in $\mathbb{K}^2$.
- A prirori estimates in $\mathbb{K}^p$.

Generalization to quadratic-exponentional RBSDEs (involving Z)

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)\, dC_s + \int_t^T dK_s - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(e) q(dsde), \quad 0 \le t \le T, \\ Y_t \ge L_t, \quad 0 \le t \le T, \\ \int_0^T (Y_{s-} - L_{s-})\, dK_s = 0, \quad \mathbb{P}\text{-a.s.} \end{cases} \tag{11}$$

With quadratic-exponential growth condition:

$$\begin{aligned} & -\left(\alpha_t + \beta|y| + \frac{\lambda}{2}|z|^2\right) dt + \left(-\alpha_t - \beta|y| - \frac{1}{\lambda} j_\lambda(t, -u)\right) dA_t \\ & \le f(t, y, z, u) dC_t \le \left(\alpha_t + \beta|y| + \frac{1}{\lambda} j_\lambda(t, u)\right) dA_t + \left(\alpha_t + \beta|y| + \frac{\lambda}{2}|z|^2\right) dt. \end{aligned} \tag{12}$$

Combining our discussion with the wellposedness of quadratic RBSDEs in Bayraktar and Yao, 2012 to conclude.

Application: American contingent claims pricing via utility maximization

- Risky asset:

$$
\mathrm{d}S_s = S_{s-}\left(b_s\,\mathrm{d}s + \sigma_s\,\mathrm{d}W_s + \int_{\mathbb{R}\setminus\{0\}} \beta_s(x)\tilde{N}_p(\,\mathrm{d}s,\,\mathrm{d}x)\right),
$$

- Exponential utility function: $U_{\tilde{\alpha}}(\cdot) := -\exp(-\tilde{\alpha}\cdot)$
- Value process:

$$
V_t^B(x) = \sup_{\pi\in A_t} \mathbb{E}\left(U_{\tilde{\alpha}}\left(x + \int_t^T \pi_s \frac{\mathrm{d}S_s}{S_{s-}} - B\right) \mid \mathcal{G}_t\right). \qquad (13)
$$

Connection with quadratic-exponential RBSDEs

$$V_t^B(x) = -\exp\left(-\tilde{\alpha}\left(x - Y_t\right)\right),$$

where Y_t is the first component of the solution (Y, Z, U) of the BSDE:

$$Y_t = B + \int_t^T f_s\left(Z_s, U_s\right) ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}\setminus\{0\}} U_s(x)\tilde{N}_p(ds, dx), \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.}$$

$$f(s, z, u) = \inf_{\tilde{\pi}\in\mathcal{C}} \left(\frac{\tilde{\alpha}}{2}\left|\tilde{\pi}\sigma_s - \left(z + \frac{\theta}{\tilde{\alpha}}\right)\right|^2 + \frac{1}{\tilde{\alpha}} j_{\tilde{\alpha}}(s, u - \tilde{\pi}\beta_s)\right) - \theta z - \frac{|\theta|^2}{2\tilde{\alpha}}.$$

There exists $\pi^* \in A_t$ (compact) satisfying:

$$\pi_s^* \in \arg\min_{\tilde{\pi}\in\mathcal{C}} \left(\frac{\tilde{\alpha}}{2}\left|\tilde{\pi}\sigma_s - \left(Z_s + \frac{\theta_s}{\tilde{\alpha}}\right)\right|^2 + \frac{1}{\tilde{\alpha}} j_{\tilde{\alpha}}(s, U_s - \tilde{\pi}\beta_s)\right).$$

Pricing for American contingent claims

- As in Rouge and El Karoui, 2000, the price of contingent claim B defined via utility function reads:

$$pr_t(B) = \inf\{y \in \mathbb{R}, V_t^B(y) \geq V_t^0(0) = -1\} = Y_t := Y_t(T, B).$$

- With early exercise payoff: $\{\xi_t, 0 \leqslant t \leqslant T\}$, the price

$$Y_t^A = \operatorname*{ess\ sup}_{\tau \in \mathcal{S}'_{t,T}} Y_t(\tau, \xi_\tau).$$

satisfies the RBSDE:

$$\begin{cases} Y_t^A = B + \int_t^T f(s, Z_s, U_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s - \int_t^T \int_{\mathbb{R}\setminus\{0\}} U_s(x)\tilde{N}_p(\mathrm{d}s, \mathrm{d}x) + \int_t^T dK_s; \\ Y_t^A \geq \xi_t; \\ \int_0^T \left(Y_{s^-}^A - \xi_s\right) dK_s = 0, \quad \mathbb{P}\text{-a.s.} \end{cases} \tag{14}$$

in which f satisfies all aforementioned assumptions.

Thank You